Journal of Fuzzy Extension and Applications

www.journal-fea.com

J. Fuzzy. Ext. Appl. Vol. 5, No. 4 (2024) 533–559.

Paper Type: Research Paper

New Operations on Cubic Intuitionistic Fuzzy Sets under P–Order

Priyadharshini Manoharan[1], Jayanthi Duraisamy[1,*], Saranya Manoharan[1]

[1] Department of Mathematics, Avinashilingam Institute for Home Science and Higher Education for Women, Tamil Nadu, India; priyadharshini1501@gmail.com; jeyanthi_mat@avinuty.ac.in; saranya22maths@gmail.com.

Citation:

Received: 16 Februray 2024 *Revised: 20 April 2024* *Accepted: 06 May 2024*	Manoharan, P., Duraisamy, J., & Manoharan, S. (2024). New operations on cubic intuitionistic fuzzy sets under P–order. *Journal of fuzzy extension and applications*, *5*(4), 533-559.

Abstract

Cubic intuitionistic Fuzzy Sets (FSs) have the efficiency to handle hybrid information, which combines interval-valued intuitionistic FSs and intuitionistic FSs. The aim is to define operators on cubic intuitionistic FSs that are capable of more effectively handling uncertain or imprecise information contained in image data. In this piece of work, few operators on the cubic intuitionistic FSs under P – order, such as arithmetic mean (@), geometric mean ($), multiplication operator (*), '#' and necessity (□) and possibility (◇) operations on the cubic intuitionistic FSs under P – order, are defined. The concept of some more modal operators like $\mathfrak{D}_{\acute{\alpha}}(\mathcal{A})$, $\mathfrak{F}_{\acute{\alpha},\acute{\beta}}(\mathcal{A})$, $\mathfrak{G}_{\acute{\alpha},\acute{\beta}}(\mathcal{A})$, $\mathfrak{H}_{\acute{\alpha},\acute{\beta}}(\mathcal{A})$, ${\mathfrak{H}^*}_{\acute{\alpha},\acute{\beta}}(\mathcal{A})$, $\mathfrak{J}_{\acute{\alpha},\acute{\beta}}(\mathcal{A})$ and ${\mathfrak{J}^*}_{\acute{\alpha},\acute{\beta}}(\mathcal{A})$ over cubic intuitionistic FSs under P – order are introduced. Additionally, the behaviour of cubic intuitionistic fuzzy modal operators is also carefully examined. Moreover, various properties related to operations on cubic intuitionistic FSs under P-order have been thoroughly studied and demonstrated.

Keywords: Intuitionistic fuzzy sets, Interval-valued intuitionistic fuzzy sets, Cubic sets, Cubic intuitionistic fuzzy sets, Modal operators.

1 | Introduction

For the first time, Zadeh [1] discussed the philosophy of Fuzzy Sets (FSs) in 1965. FS theory has been applied to characterize imprecise or nebulous situations. As an extension of FSs, Zadeh [2] created the concept of Interval-Valued Fuzzy Sets (IVFSs) in 1975. IVFSs can be characterized by the membership degree of an interval value between 0 and 1. In 1986, Atanassov [3] published the first description of Intuitionistic Fuzzy Sets (IFSs), a generalization of FSs that contain degrees of membership and non-membership.

Corresponding Author: jeyanthi_mat@avinuty.ac.in

https://doi.org/10.22105/jfea.2024.446248.1393

In 1989, Gorzalczany [4] made certain inference techniques for IVFSs and their properties public. Atanassov and Gargov [5] provided the first description of Interval-Valued Intuitionistic Fuzzy Sets (IVIFS) in 1989. For the theory of IFSs, Atanassov [6–12] defined numerous Operators (OPs) in various years. In 1994, Atanassov [13] presented a few OPs over IVIFSs. Moreover, Atanassov [14] presented some modal OPs in 2018 over IVIFSs.

Over an IVIFS of the second type, Rajesh and Srinivasan [15], [16] defined a few OPs and established their features. In 2014, Sharma [17] investigated how IF groups are affected by the modal OP $F_{(\alpha,\beta)}(A)$. Then several extensions of the OPs were studied by different authors [18–22] in various years, and some properties were verified [23], [24] in 2008 and 2017.

The average OPs were first introduced by Bhattacharya [25–27], and several novel properties in IFSs were created with the aid of specific OPs and modal OPs in 2016 and 2021. Iqbal [28] described new topological OPs, modal-like OPs, and negation over IFSs of cube root type in 2019.

In 2012, Jun et al. [29] introduced Cubic Sets (CSs), which fuse IVFSs with FSs. They divided CSs into two categories: internal and external. They classified CSs as internal and external CSs. In addition to the operations P(R)-union and P(R)-intersection, they examined the concepts of P(R)-order CSs. Notably, the record for CSs does not include a non-membership degree corresponding to a membership degree. In the same year, Wang et al. [30] examined various definitions of aggregation OPs for IVIFS and applied the developed OPs to Multi-Criteria Decision-Making Problems (MCDMPs) for prioritizing investment projects.

Sudharsan and Ezhlimaran [31], [32] introduced two new OPs based on IFSs and IVIFSs and proved several theorems in 2014 and 2015. In 2017, Jamkhaneh [33] defined four OPs over-generalized IVIFSs and discussed some basic properties of these new OPs. In the same year, Mahmood et al. [34] presented MCDMP-based cubic aggregation OPs. In 2017, Tarsuslu et al. [35] defined new model OPs over the IFSs, which were undefined by Atanasov [36] and identified some of their characteristics.

A CS is generalized to a Cubic Intuitionistic Fuzzy Set (CIFS). Kaur and Garg [37] introduced CIFSs for the first time in 2018. This concept captures both membership and non-membership degrees. They explored the concepts of P-order and R-order in CIFSs and the operations of union, intersection, addition, and product within these orders. Additionally, they classified CIFSs into internal and external categories. A CIFS is an effective tool for handling complex decision-making scenarios. In the same year, Fitting [38] introduced necessity and possibility OPs over IFSs and provided several theorems using the proposed OPs.

In 2019, Chaira [39] discussed fuzzy and IF OPs and various types of aggregation OPs and applied them to MCDMP. Rashid et al. [40] defined N-CSs and introduced several types of OPs, including N-C aggregate OPs and union and intersection operations under P-order and R-order. Additionally, Wang and Mendel [41] developed a prioritized arithmetic mean in 2019 to address IVIF MCDM scenarios. They also investigated some of its desirable properties in detail.

In 2020, Muneeza and Abdullah [42] established the concept of Intuitionistic Cubic Fuzzy Sets (ICFSs) and defined several OPs for ICFSs. They also established a series of weighted aggregation OPs and applied these OPs to supply chain management. A novel OP Δ over IFSs was introduced in 2021, and some of its features were examined by Atanassova and Dworniczak [43]. In 2022, Fan et al. [44] proposed several new IVIF aggregation OPs based on new IVIF OP rules, combining IVIFS theory with dynamic MCDMP.

The literature on FSs and their extensions is extensive and frequently updated. Several significant works contribute to this field. In 2021, Abbasi Shureshjani and Shakouri [45] provided comments on a novel parametric ranking method for IF numbers. Also, in the same year, Akram et al. [46] introduced a hybrid decision-making analysis using complex q-rung picture fuzzy Einstein averaging OPs. Additionally, in 2021, Dehghani Filabadi and Hesamian [47] developed a multi-period MCDM using type-2 FSs of linguistic variables, demonstrating the ability of FSs in decision-making processes. Further, Akram et al. [48] addressed the fractional transportation problem under IV fermatean FSs, highlighting the practical applications of FSs in transportation and logistics in 2022.

In 2022 and 2023, Bhattacharya [49–51] established several new relationships between OPs and demonstrated significant equalities using modal OPs on IFSs. In order to verify the efficacy and legitimacy of the suggested OPs, they showcased an application built on MCDMP.

In the same year, Liu et al. [52] introduced new OPs based on ICFSs and two MCDMPs utilizing these OPs. They applied the proposed decision models to supplier selection and verified the superior reliability and accuracy of their methods through comparative analysis with existing approaches. In 2023, Additionally, Nagarajan et al. [53] proposed a novel approach based on Neutrosophic (NT) Bonferroni mean OP for MCDMP in NT interval environments.

In 2024, Adak et al. [54] proposed several new OPs on Pythagorean FSs. Innovative approaches continue to emerge, such as Ali et al.'s [55] application of Yao's three-way decision model using IFSs for medical diagnosis in 2024. In the same year, Bakbak et al. [56] explored MCDM applications based on trapezoidal fuzzy multi-number preference relations in architectural contexts. Recent studies by Jameel and Tanwar [57] in 2024 introduced fuzzy MCDM to address climate change adaptation planning barriers, while Mostafa et al. [58] conducted a comparative study on X-ray image enhancement using NTS.

In 2024, Rasoulzadeh et al. [59] presented a hybrid model for selecting optimal stock portfolios under IFSs, showcasing the financial applications of FSs. Finally, in the same year, Ghanizadeh [60] evaluated human resources using a GRA-based decision model in an IVF environment, contributing to the fields of decision and operations research.

Various authors [61-73] have used different fuzzy extensions such as Pythagorean FSs, fermatean FSs, probabilistic hesitant FSs, IV fermatean FSs, orthopair multi-FSs, hesitant FSs, IFSs, soft sets, hyper-soft sets and CIFSs in various fields in recent years. One of the main extensions of FSs is NTSs [74–79], which are used in multi-objective linear fractional programming problems, MCDMPs, and solving Gaussian-valued NT shortest path problems.

Numerous studies have explored various OPs for IFSs and IVIFSs. However, few articles have studied the OPs of a CIFS. Motivated by the advantages of CIFSs, this article proposes the OPs of CIFSs based on IVIFSs and IFSs. These OPs are capable of modeling ambiguous image regions or elements that cannot be definitively classified, therefore contributing to the extraction and recognition of more accurate features. Also, this study will demonstrate an interest in using CIF OPs in the context of information science and knowledge systems.

The remaining sections of this article are organized as follows. In Section 2, some fundamental ideas concerning the CIFSs and their OPs were reviewed. The idea of operations on IVIFSs and IFSs is expanded to include operations on CIFSs in Section 3. The major reason for expanding this strategy is to get a better outcome, which confirms several of its characteristics. In Section 4, we described how to solve the MCDMP using the OPs and provided an example to clarify the process.

2 | Preliminaries

This section provides definitions extensively utilized in this paper.

Definition 1 ([80]). A CIFS $\breve{\mathfrak{A}}$ defined over the Universal Set (US) $\mathfrak{S}$ is an ordered pair which is defined as follows:

$$\breve{\mathfrak{A}} = \{\langle \overset{\frown}{\mathfrak{s}}, \ddot{\mathbb{A}}(\overset{\frown}{\mathfrak{s}}), \varkappa(\overset{\frown}{\mathfrak{s}})\rangle | \overset{\frown}{\mathfrak{s}} \in \mathfrak{S}\},$$

where $\ddot{\mathbb{A}} = \{\langle \overset{\frown}{\mathfrak{s}}, [\eth^-(\overset{\frown}{\mathfrak{s}}), \eth^+(\overset{\frown}{\mathfrak{s}})], [\xi^-(\overset{\frown}{\mathfrak{s}}), \xi^+(\overset{\frown}{\mathfrak{s}})]\rangle | \overset{\frown}{\mathfrak{s}} \in \mathfrak{S}\}$ represents the IVIFS defined on $\mathfrak{S}$ while $\varkappa = \{\overset{\frown}{\mathfrak{s}}, \langle \eth(\overset{\frown}{\mathfrak{s}}), \xi(\overset{\frown}{\mathfrak{s}})\rangle | \overset{\frown}{\mathfrak{s}} \in \mathfrak{S}\}$ represents an IFS such that $0 \leq \eth^-(\overset{\frown}{\mathfrak{s}}) \leq \eth^+(\overset{\frown}{\mathfrak{s}}) \leq 1$, $0 \leq \xi^-(\overset{\frown}{\mathfrak{s}}) \leq \xi^+(\overset{\frown}{\mathfrak{s}}) \leq 1$, and $0 \leq \eth^+(\overset{\frown}{\mathfrak{s}}) + \xi^+(\overset{\frown}{\mathfrak{s}}) \leq 1$. Also, $0 \leq \eth(\overset{\frown}{\mathfrak{s}}), \xi(\overset{\frown}{\mathfrak{s}}) \leq 1$ and $\eth(\overset{\frown}{\mathfrak{s}}) + \xi(\overset{\frown}{\mathfrak{s}}) \leq 1$. For the sake of simplicity, we denote these pairs as $\breve{\mathfrak{A}} = \langle \ddot{\mathbb{A}}, \varkappa\rangle$, where $\ddot{\mathbb{A}} = \langle [\eth^-, \eth^+], [\xi^-, \xi^+]\rangle$ and $\varkappa = \langle \eth, \xi\rangle$ and call them the Cubic Intuitionistic Fuzzy Number (CIFN).

Definition 2 ([80]). Let $\breve{\mathfrak{A}} = \langle \ddot{A}, \varkappa \rangle$ and $\breve{\mathfrak{B}} = \langle \ddot{\mathbb{B}}, \upsilon \rangle$ represents two CIFSs in $\mathfrak{S}$. Next, we have

P - order: $\breve{\mathfrak{A}} \subseteq_P \breve{\mathfrak{B}} \Leftrightarrow \left[ð_{\breve{\mathfrak{A}}}^{-}, ð_{\breve{\mathfrak{A}}}^{+}\right] \subseteq \left[ð_{\breve{\mathfrak{B}}}^{-}, ð_{\breve{\mathfrak{B}}}^{+}\right], \left[\xi_{\breve{\mathfrak{A}}}^{-}, \xi_{\breve{\mathfrak{A}}}^{+}\right] \supseteq \left[\xi_{\breve{\mathfrak{B}}}^{-}, \xi_{\breve{\mathfrak{B}}}^{+}\right], ð_{\breve{\mathfrak{A}}} \leq ð_{\breve{\mathfrak{B}}}$ and $\xi_{\breve{\mathfrak{A}}} \geq \xi_{\breve{\mathfrak{B}}}$.

Definition 3 ([80]). A CIFS $\breve{\mathfrak{A}}$ is said to be internal CIFS if $ð^{-}(\breve{\mathfrak{s}}) \leq ð(\breve{\mathfrak{s}}) \leq ð^{+}(\breve{\mathfrak{s}})$ and $\xi^{-}(\breve{\mathfrak{s}}) \leq \xi(\breve{\mathfrak{s}}) \leq \xi^{+}(\breve{\mathfrak{s}})$ for all $\breve{\mathfrak{s}} \in \mathfrak{S}$.

Definition 4 ([80]). A CIFS $\breve{\mathfrak{A}}$ is said to be external CIFS if $ð(\breve{\mathfrak{s}}) \notin \left[ð^{-}(\breve{\mathfrak{s}}), ð^{+}(\breve{\mathfrak{s}})\right]$ and $\xi(\breve{\mathfrak{s}}) \notin \left[\xi^{-}(\breve{\mathfrak{s}}), \xi^{+}(\breve{\mathfrak{s}})\right]$ for all $\breve{\mathfrak{s}} \in \mathfrak{S}$.

3 | Operations on Cubic Intuitionistic Fuzzy Sets Under P – order

Following Atanassov [9], [13], this part of the article introduced novel operations on CIFSs under P-order over the US $\mathfrak{S}$ and investigated few of their features.

Let $\breve{\mathfrak{A}}$ and $\breve{\mathfrak{B}}$ be CIFSs on the US $\mathfrak{S}$, where

$$\breve{\mathfrak{A}} = \left\{\left(\breve{\mathfrak{s}}, \langle [ð_{\breve{\mathfrak{A}}}^{-}(\breve{\mathfrak{s}}), ð_{\breve{\mathfrak{A}}}^{+}(\breve{\mathfrak{s}})], [\xi_{\breve{\mathfrak{A}}}^{-}(\breve{\mathfrak{s}}), \xi_{\breve{\mathfrak{A}}}^{+}(\breve{\mathfrak{s}})] \rangle, \langle ð_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}), \xi_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) \rangle \right) \middle| \breve{\mathfrak{s}} \in \mathfrak{S}\right\},$$

and

$$\breve{\mathfrak{B}} = \left\{\left(\breve{\mathfrak{s}}, \langle [ð_{\breve{\mathfrak{B}}}^{-}(\breve{\mathfrak{s}}), ð_{\breve{\mathfrak{B}}}^{+}(\breve{\mathfrak{s}})], [\xi_{\breve{\mathfrak{B}}}^{-}(\breve{\mathfrak{s}}), \xi_{\breve{\mathfrak{B}}}^{+}(\breve{\mathfrak{s}})] \rangle, \langle ð_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}}), \xi_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}}) \rangle \right) \middle| \breve{\mathfrak{s}} \in \mathfrak{S}\right\}.$$

Definition 5. Let $\breve{\mathfrak{A}}$ and $\breve{\mathfrak{B}}$ are CIFSs of the set $\mathfrak{S}$, then the arithmetic mean OP is defined as follows:

$$\breve{\mathfrak{A}} \,@\, \breve{\mathfrak{B}} = \left\{\left(\begin{matrix}\breve{\mathfrak{s}}, \left\langle \left[\frac{ð_{\breve{\mathfrak{A}}}^{-}(\breve{\mathfrak{s}})+ð_{\breve{\mathfrak{B}}}^{-}(\breve{\mathfrak{s}})}{2}, \frac{ð_{\breve{\mathfrak{A}}}^{+}(\breve{\mathfrak{s}})+ð_{\breve{\mathfrak{B}}}^{+}(\breve{\mathfrak{s}})}{2}\right], \left[\frac{\xi_{\breve{\mathfrak{A}}}^{-}(\breve{\mathfrak{s}})+\xi_{\breve{\mathfrak{B}}}^{-}(\breve{\mathfrak{s}})}{2}, \frac{\xi_{\breve{\mathfrak{A}}}^{+}(\breve{\mathfrak{s}})+\xi_{\breve{\mathfrak{B}}}^{+}(\breve{\mathfrak{s}})}{2}\right]\right\rangle, \\ \left\langle \frac{ð_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}})+ð_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}})}{2}, \frac{\xi_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}})+\xi_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}})}{2} \right\rangle\end{matrix}\right) \middle| \breve{\mathfrak{s}} \in \mathfrak{S}\right\}.$$

Definition 6. Let $\breve{\mathfrak{A}}$ and $\breve{\mathfrak{B}}$ are CIFSs of the set $\mathfrak{S}$, then the geometric mean OP is defined as follows:

$$\breve{\mathfrak{A}} \,\$\, \breve{\mathfrak{B}} = \left\{\left(\begin{matrix}\breve{\mathfrak{s}}, \left\langle \left[\sqrt{ð_{\breve{\mathfrak{A}}}^{-}(\breve{\mathfrak{s}}) \,.\, ð_{\breve{\mathfrak{B}}}^{-}(\breve{\mathfrak{s}})}, \sqrt{ð_{\breve{\mathfrak{A}}}^{+}(\breve{\mathfrak{s}}) \,.\, ð_{\breve{\mathfrak{B}}}^{+}(\breve{\mathfrak{s}})}\right], \left[\sqrt{\xi_{\breve{\mathfrak{A}}}^{-}(\breve{\mathfrak{s}}) \,.\, \xi_{\breve{\mathfrak{B}}}^{-}(\breve{\mathfrak{s}})}, \sqrt{\xi_{\breve{\mathfrak{A}}}^{+}(\breve{\mathfrak{s}}) \,.\, \xi_{\breve{\mathfrak{B}}}^{+}(\breve{\mathfrak{s}})}\right]\right\rangle, \\ \left\langle \sqrt{ð_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) \,.\, ð_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}})}, \sqrt{\xi_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) \,.\, \xi_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}})} \right\rangle\end{matrix}\right) \middle| \breve{\mathfrak{s}} \in \mathfrak{S}\right\}.$$

Definition 7. Let $\breve{\mathfrak{A}}$ and $\breve{\mathfrak{B}}$ are CIFSs of the set $\mathfrak{S}$, then the multiplication OP is defined as follows:

$$\breve{\mathfrak{A}} * \breve{\mathfrak{B}} =$$

$$\left\{\left(\begin{matrix}\breve{\mathfrak{s}}, \left\langle \left[\frac{ð_{\breve{\mathfrak{A}}}^{-}(\breve{\mathfrak{s}})+ð_{\breve{\mathfrak{B}}}^{-}(\breve{\mathfrak{s}})}{2\left(ð_{\breve{\mathfrak{A}}}^{-}(\breve{\mathfrak{s}}).ð_{\breve{\mathfrak{B}}}^{-}(\breve{\mathfrak{s}})+1\right)}, \frac{ð_{\breve{\mathfrak{A}}}^{+}(\breve{\mathfrak{s}})+ð_{\breve{\mathfrak{B}}}^{+}(\breve{\mathfrak{s}})}{2\left(ð_{\breve{\mathfrak{A}}}^{+}(\breve{\mathfrak{s}}).ð_{\breve{\mathfrak{B}}}^{+}(\breve{\mathfrak{s}})+1\right)}\right], \left[\frac{\xi_{\breve{\mathfrak{A}}}^{-}(\breve{\mathfrak{s}})+\xi_{\breve{\mathfrak{B}}}^{-}(\breve{\mathfrak{s}})}{2\left(\xi_{\breve{\mathfrak{A}}}^{-}(\breve{\mathfrak{s}}).\xi_{\breve{\mathfrak{B}}}^{-}(\breve{\mathfrak{s}})+1\right)}, \frac{\xi_{\breve{\mathfrak{A}}}^{+}(\breve{\mathfrak{s}})+\xi_{\breve{\mathfrak{B}}}^{+}(\breve{\mathfrak{s}})}{2\left(\xi_{\breve{\mathfrak{A}}}^{+}(\breve{\mathfrak{s}}).\xi_{\breve{\mathfrak{B}}}^{+}(\breve{\mathfrak{s}})+1\right)}\right]\right\rangle, \\ \left\langle \frac{ð_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}})+ð_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}})}{2\left(ð_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}).ð_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}})+1\right)}, \frac{\xi_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}})+\xi_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}})}{2\left(\xi_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}).\xi_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}})+1\right)} \right\rangle\end{matrix}\right) \middle| \breve{\mathfrak{s}} \in \mathfrak{S}\right\}.$$

Definition 8. Let $\breve{\mathfrak{A}}$ and $\breve{\mathfrak{B}}$ are CIFSs of the set $\mathfrak{S}$, then $\breve{\mathfrak{A}} \,\#\, \breve{\mathfrak{B}}$ is defined as follows:

$$\breve{\mathfrak{A}} \,\#\, \breve{\mathfrak{B}} =$$

$$\left\{\left(\begin{matrix}\breve{\mathfrak{s}}, \left\langle \left[\frac{2\, ð_{\breve{\mathfrak{A}}}^{-}(\breve{\mathfrak{s}}).ð_{\breve{\mathfrak{B}}}^{-}(\breve{\mathfrak{s}})}{\left(ð_{\breve{\mathfrak{A}}}^{-}(\breve{\mathfrak{s}})+ð_{\breve{\mathfrak{B}}}^{-}(\breve{\mathfrak{s}})\right)}, \frac{2\, ð_{\breve{\mathfrak{A}}}^{+}(\breve{\mathfrak{s}}).ð_{\breve{\mathfrak{B}}}^{+}(\breve{\mathfrak{s}})}{\left(ð_{\breve{\mathfrak{A}}}^{+}(\breve{\mathfrak{s}})+ð_{\breve{\mathfrak{B}}}^{+}(\breve{\mathfrak{s}})\right)}\right], \left[\frac{2\, \xi_{\breve{\mathfrak{A}}}^{-}(\breve{\mathfrak{s}}).\xi_{\breve{\mathfrak{B}}}^{-}(\breve{\mathfrak{s}})}{\left(\xi_{\breve{\mathfrak{A}}}^{-}(\breve{\mathfrak{s}})+\xi_{\breve{\mathfrak{B}}}^{-}(\breve{\mathfrak{s}})\right)}, \frac{2\, \xi_{\breve{\mathfrak{A}}}^{+}(\breve{\mathfrak{s}}).\xi_{\breve{\mathfrak{B}}}^{+}(\breve{\mathfrak{s}})}{\left(\xi_{\breve{\mathfrak{A}}}^{+}(\breve{\mathfrak{s}})+\xi_{\breve{\mathfrak{B}}}^{+}(\breve{\mathfrak{s}})\right)}\right]\right\rangle, \\ \left\langle \frac{2\, ð_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}).ð_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}})}{\left(ð_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}})+ð_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}})\right)}, \frac{2\, \xi_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}})\,.\xi_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}})}{\left(\xi_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}})+\xi_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}})\right)} \right\rangle\end{matrix}\right) \middle| \breve{\mathfrak{s}} \in \mathfrak{S}\right\}.$$

For which we shall accept that

If $ð^-_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) = ð^-_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}}) = 0$ then $\frac{ð^-_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}})\,.\,ð^-_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}})}{\left(ð^-_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) + ð^-_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}})\right)} = 0$, If $ð^+_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) = ð^+_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}}) = 0$ then $\frac{ð^+_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}})\,.\,ð^+_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}})}{\left(ð^+_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) + ð^+_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}})\right)} = 0$.

If $\xi^-_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) = \xi^-_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}}) = 0$ then $\frac{\xi^-_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}})\,.\,\xi^-_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}})}{\left(\xi^-_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) + \xi^-_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}})\right)} = 0$, If $\xi^+_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) = \xi^+_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}}) = 0$ then $\frac{\xi^+_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}})\,.\,\xi^+_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}})}{\left(\xi^+_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) + \xi^+_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}})\right)} = 0$.

If $ð_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) = ð_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}}) = 0$ then $\frac{ð_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}})\,.\,ð_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}})}{\left(ð_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) + ð_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}})\right)} = 0$, If $\xi_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) = \xi_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}}) = 0$ then $\frac{\xi_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}})\,.\,\xi_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}})}{\left(\xi_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) + \xi_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}})\right)} = 0$.

Example 1. Consider the US $\mathfrak{S} = \{\breve{\mathfrak{s}}_1, \breve{\mathfrak{s}}_2, \breve{\mathfrak{s}}_3, \breve{\mathfrak{s}}_4, \breve{\mathfrak{s}}_5\}$. Let $\breve{\mathfrak{A}}$ and $\breve{\mathfrak{B}}$ be two CIFSs of $\mathfrak{S}$ given by

$\breve{\mathfrak{A}} = \{(\langle[0.1,0.5], [0.1,0.4]\rangle, \langle 0.5,0.1\rangle), (\langle[0.4,0.5], [0.2,0.3]\rangle, \langle 0.3, 0.2\rangle),$

$(\langle[0.4,0.5], [0.1,0.2]\rangle, \langle 0.5,0.3\rangle), (\langle[0.2,0.4], [0.2,0.2]\rangle, \langle 0.1, 0.7\rangle), (\langle[0.4,0.5], [0.4,0.4]\rangle, \langle 0.2, 0.3\rangle)\}.$

$\breve{\mathfrak{B}} = \{(\langle[0.2,0.5], [0.4,0.5]\rangle, \langle 0.5, 0.5\rangle), (\langle[0.3,0.4], [0.3,0.4]\rangle, \langle 0.6,0.1\rangle),$

$(\langle[0.1,0.5], [0.2,0.3]\rangle, \langle 0.5,0.1\rangle), (\langle[0.4,0.5], [0.1,0.4]\rangle, \langle 0.4,0.2\rangle), (\langle[0.1,0.3], [0.2,0.5]\rangle, \langle 0.4,0.5\rangle)\}.$

From the *Definition 5*, we have

$\frac{ð^-_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}_1)+ð^-_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}}_1)}{2} = \frac{0.1+0.2}{2} = 0.15$, $\frac{ð^+_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}_1)+ð^+_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}}_1)}{2} = \frac{0.5+0.5}{2} = 0.50$, $\frac{\xi^-_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}_1)+\xi^-_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}}_1)}{2} =$
$\frac{0.1+0.4}{2} = 0.25$, $\frac{\xi^+_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}_1)+\xi^+_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}}_1)}{2} = \frac{0.4+0.5}{2} = 0.45$, $\frac{ð_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}_1)+ð_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}}_1)}{2} = \frac{0.5+0.5}{2} = 0.50$,
$\frac{\xi_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}_1)+\xi_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}}_1)}{2} = \frac{0.1+0.5}{2} = 0.30.$

Hence we obtain

$\breve{\mathfrak{A}}\ @\ \breve{\mathfrak{B}} = \{(\langle[0.15,0.50], [0.25, 0.45]\rangle, \langle 0.50,0.30\rangle)\}.$

From the *Definition 6*, we have

$\sqrt{ð^-_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}_1)\,.\,ð^-_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}}_1)} = \sqrt{(0.1)\,.\,(0.2)} = 0.14$, $\sqrt{ð^+_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}_1)\,.\,ð^+_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}}_1)} = \sqrt{(0.5)\,.\,(0.5)} =$
0.50,

$\sqrt{\xi^-_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}_1)\,.\,\xi^-_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}}_1)} = \sqrt{(0.1)\,.\,(0.4)} = 0.20$, $\sqrt{\xi^+_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}_1)\,.\,\xi^+_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}}_1)} = \sqrt{(0.4)\,.\,(0.5)} =$
0.45,

$\sqrt{ð_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}_1)\,.\,ð_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}}_1)} = \sqrt{(0.5)\,.\,(0.5)} = 0.50$ and $\sqrt{\xi_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}_1)\,.\,\xi_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}}_1)} = \sqrt{(0.1)\,.\,(0.5)} =$
0.22.

Hence we obtain

$\breve{\mathfrak{A}}\ \$\ \breve{\mathfrak{B}} = \{(\langle[0.14,0.50], [0.20, 0.45]\rangle, \langle 0.50,0.22\rangle)\}.$

From the *Definition 7*, we have

$\frac{ð^-_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}_1)+ð^-_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}}_1)}{2\left(ð^-_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}_1).ð^-_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}}_1)+1\right)} = \frac{0.1+0.2}{2((0.1)\,.\,(0.2)+1)} = \frac{0.3}{2.04} = 0.15$, $\frac{ð^+_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}_1)+ð^+_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}}_1)}{2\left(ð^+_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}_1).ð^+_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}}_1)+1\right)} =$
$\frac{0.5+0.5}{2((0.5)\,.(0.5)+1)} = \frac{0.10}{2.5} = 0.40,$

$$\frac{\xi^{-}_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}_1)+\xi^{-}_{\breve{\mathfrak{B}}}(\widetilde{\mathfrak{s}}_1)}{2\left(\xi^{-}_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}_1).\xi^{-}_{\breve{\mathfrak{B}}}(\widetilde{\mathfrak{s}}_1)+1\right)}=\frac{0.1+0.4}{2((0.1)\,.\,(0.4)+1)}=\frac{0.5}{2.08}=0.24,\ \frac{\xi^{+}_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}_1)+\xi^{+}_{\breve{\mathfrak{B}}}(\widetilde{\mathfrak{s}}_1)}{2\left(\xi^{+}_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}_1).\xi^{+}_{\breve{\mathfrak{B}}}(\widetilde{\mathfrak{s}}_1)+1\right)}=\frac{0.4+0.5}{2((0.4)\,.\,(0.5)+1)}=\frac{0.9}{2.4}=0.38,$$

$$\frac{\eth_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}_1)+\eth_{\breve{\mathfrak{B}}}(\widetilde{\mathfrak{s}}_1)}{2(\eth_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}_1).\eth_{\breve{\mathfrak{B}}}(\widetilde{\mathfrak{s}}_1)+1)}=\frac{0.5+0.5}{2((0.5)\,.\,(0.5)+1)}=\frac{0.10}{2.5}=0.40 \text{ and } \frac{\xi_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}_1)+\xi_{\breve{\mathfrak{B}}}(\widetilde{\mathfrak{s}}_1)}{2(\xi_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}_1).\xi_{\breve{\mathfrak{B}}}(\widetilde{\mathfrak{s}}_1)+1)}=\frac{0.1+0.5}{2((0.1)\,.\,(0.5)+1)}=\frac{0.6}{2.1}=0.29.$$

Hence we obtain

$$\breve{\mathfrak{A}} * \breve{\mathfrak{B}} = \{(\langle[0.15,0.40],[0.24,0.38]\rangle,\langle 0.40,0.29\rangle)\}.$$

From the *Definition 8*, we have

$$\frac{2\,\eth^{-}_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}_1).\eth^{-}_{\breve{\mathfrak{B}}}(\widetilde{\mathfrak{s}}_1)}{\left(\eth^{-}_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}_1)+\eth^{-}_{\breve{\mathfrak{B}}}(\widetilde{\mathfrak{s}}_1)\right)}=\frac{(2)\,.(\,0.1)\,.(\,0.2)}{(0.1+0.2)}=\frac{0.04}{0.3}=0.13,\ \frac{2\,\eth^{+}_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}_1).\eth^{+}_{\breve{\mathfrak{B}}}(\widetilde{\mathfrak{s}}_1)}{\left(\eth^{+}_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}_1)+\eth^{+}_{\breve{\mathfrak{B}}}(\widetilde{\mathfrak{s}}_1)\right)}=\frac{(2\,).\ (0.5)\,.\ (0.5)}{(0.5+0.5)}=\frac{0.5}{0.10}=0.50,$$

$$\frac{2\,\xi^{-}_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}_1).\xi^{-}_{\breve{\mathfrak{B}}}(\widetilde{\mathfrak{s}}_1)}{\left(\xi^{-}_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}_1)+\xi^{-}_{\breve{\mathfrak{B}}}(\widetilde{\mathfrak{s}}_1)\right)}=\frac{(2\,).\ (0.1)\,.\ (0.4)}{(0.1+0.4)}=\frac{0.08}{0.5}=0.16,\frac{2\,\xi^{+}_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}_1).\xi^{+}_{\breve{\mathfrak{B}}}(\widetilde{\mathfrak{s}}_1)}{\left(\xi^{+}_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}_1)+\xi^{+}_{\breve{\mathfrak{B}}}(\widetilde{\mathfrak{s}}_1)\right)}=\frac{(2)\,.\ (0.4)\,.\ (0.5)}{(0.4+0.5)}=\frac{0.4}{0.9}=0.44$$

$$\frac{2\,\eth_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}_1).\eth_{\breve{\mathfrak{B}}}(\widetilde{\mathfrak{s}}_1)}{\left(\eth_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}_1)+\eth_{\breve{\mathfrak{B}}}(\widetilde{\mathfrak{s}}_1)\right)}=\frac{(2)\,.\ (0.5)\,.\ (0.5)}{(0.5+0.5)}=\frac{0.5}{0.10}=0.50 \text{ and } \frac{2\,\xi_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}_1)\,.\xi_{\breve{\mathfrak{B}}}(\widetilde{\mathfrak{s}}_1)}{\left(\xi_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}_1)+\xi_{\breve{\mathfrak{B}}}(\widetilde{\mathfrak{s}}_1)\right)}=\frac{(2)\,.\ (0.1)\,.\ (0.5)}{(0.1+0.5)}=\frac{0.1}{0.6}=0.17.$$

Hence we obtain

$$\breve{\mathfrak{A}} \# \breve{\mathfrak{B}} = \{(\langle[0.13,0.50],[0.16,0.44]\rangle,\langle 0.50,0.17\rangle)\}.$$

Similarly, the OPs for other elements are obtained and presented in the below table.

Table 1. The OPs on CIFSs.

Operator	$\widetilde{\mathfrak{s}}_1$	$\widetilde{\mathfrak{s}}_2$	$\widetilde{\mathfrak{s}}_3$	$\widetilde{\mathfrak{s}}_4$	$\widetilde{\mathfrak{s}}_5$
$\breve{\mathfrak{A}} @ \breve{\mathfrak{B}}$	$\langle[0.15,0.5],[0.25,0.45]\rangle$, $\langle 0.5,0.3\rangle$	$\langle[0.35,0.45],[0.25,0.35]\rangle$, $\langle 0.45, 0.15\rangle$	$\langle[0.25,0.5],[0.15,0.25]\rangle$, $\langle 0.5,0.2\rangle$	$\langle[0.3,0.45],[0.15,0.3]\rangle$, $\langle 0.25, 0.45\rangle$	$\langle[0.25,0.4],[0.3,0.45]\rangle$, $\langle 0.3, 0.4\rangle$
$\breve{\mathfrak{A}} \$ \breve{\mathfrak{B}}$	$\langle[0.14,0.5],[0.2,0.45]\rangle$, $\langle 0.5,0.22\rangle$	$\langle[0.35,0.45],[0.24,0.35]\rangle$, $\langle 0.42, 0.14\rangle$	$\langle[0.2,0.5],[0.14,0.24]\rangle$, $\langle 0.5,0.17\rangle$	$\langle[0.28,0.45],[0.14,0.28]\rangle$, $\langle 0.2, 0.37\rangle$	$\langle[0.2,0.39],[0.28,0.45]\rangle$, $\langle 0.28, 0.39\rangle$
$\breve{\mathfrak{A}} * \breve{\mathfrak{B}}$	$\langle[0.15,0.4],[0.24,0.38]\rangle$, $\langle 0.4,0.29\rangle$	$\langle[0.31,0.38],[0.24,0.31]\rangle$, $\langle 0.38, 0.15\rangle$	$\langle[0.24,0.4],[0.15,0.24]\rangle$, $\langle 0.4,0.19\rangle$	$\langle[0.28,0.38],[0.15,0.28]\rangle$, $\langle 0.24, 0.39\rangle$	$\langle[0.24,0.35],[0.28,0.38]\rangle$, $\langle 0.28, 0.35\rangle$
$\breve{\mathfrak{A}} \# \breve{\mathfrak{B}}$	$\langle[0.13,0.5],[0.$ $\langle 0.5,0.17\rangle$	$\langle[0.34,0.44],[0.$ $\langle 0.4, 0.13\rangle$	$\langle[0.16,0.5],[0.$ $\langle 0.5,0.15\rangle$	$\langle[0.27,0.44],[$ $\langle 0.16, 0.31\rangle$	$\langle[0.16,0.38],[$ $\langle 0.27, 0.38\rangle$

It is to be noted that in *Example 1*, $\breve{\mathfrak{A}} @ \breve{\mathfrak{B}}$, $\breve{\mathfrak{A}} \$ \breve{\mathfrak{B}}$, $\breve{\mathfrak{A}} * \breve{\mathfrak{B}}$ and $\breve{\mathfrak{A}} \# \breve{\mathfrak{B}}$ are CIFSs.

Proposition 1. For any two CIFSs $\breve{\mathfrak{A}}$ and $\breve{\mathfrak{B}}$.

I. $\breve{\mathfrak{A}} @ \breve{\mathfrak{B}} = \breve{\mathfrak{B}} @ \breve{\mathfrak{A}}$.

II. $\breve{\mathfrak{A}} \$ \breve{\mathfrak{B}} = \breve{\mathfrak{B}} \$ \breve{\mathfrak{A}}$.

III. $\breve{\mathfrak{A}} * \breve{\mathfrak{B}} = \breve{\mathfrak{B}} * \breve{\mathfrak{A}}$.

IV. $\breve{\mathfrak{A}} \# \breve{\mathfrak{B}} = \breve{\mathfrak{B}} \# \breve{\mathfrak{A}}$.

V. $\overline{\bar{\bar{\mathfrak{A}}} @ \bar{\bar{\mathfrak{B}}}} = \breve{\mathfrak{A}} @ \breve{\mathfrak{B}}$.

VI. $\overline{\bar{\bar{\mathfrak{A}}} \$ \bar{\bar{\mathfrak{B}}}} = \breve{\mathfrak{A}} \$ \breve{\mathfrak{B}}$.

VII. $\overline{\bar{\bar{\mathfrak{A}}} * \bar{\bar{\mathfrak{B}}}} = \breve{\mathfrak{A}} * \breve{\mathfrak{B}}$.

VIII. $\overline{\bar{\bar{\mathfrak{A}}} \# \bar{\bar{\mathfrak{B}}}} = \breve{\mathfrak{A}} \# \breve{\mathfrak{B}}$.

IX. $(\breve{\mathfrak{A}} \cap_P \breve{\mathfrak{B}}) @ \breve{\mathfrak{C}} = (\breve{\mathfrak{A}} @ \breve{\mathfrak{C}}) \cap_P (\breve{\mathfrak{B}} @ \breve{\mathfrak{C}})$.

X. $(\breve{\mathfrak{A}} \cup_P \breve{\mathfrak{B}}) @ \breve{\mathfrak{C}} = (\breve{\mathfrak{A}} @ \breve{\mathfrak{C}}) \cup_P (\breve{\mathfrak{B}} @ \breve{\mathfrak{C}})$.

XI. $(\breve{\mathfrak{A}} \cap_P \breve{\mathfrak{B}}) \$ \breve{\mathfrak{C}} = (\breve{\mathfrak{A}} \$ \breve{\mathfrak{C}}) \cap_P (\breve{\mathfrak{B}} \$ \breve{\mathfrak{C}})$.

XII. $(\breve{\mathfrak{A}} \cup_P \breve{\mathfrak{B}}) \$ \breve{\mathfrak{C}} = (\breve{\mathfrak{A}} \$ \breve{\mathfrak{C}}) \cup_P (\breve{\mathfrak{B}} \$ \breve{\mathfrak{C}})$.

XIII. $(\breve{\mathfrak{A}} \cap_P \breve{\mathfrak{B}}) * \breve{\mathfrak{C}} = (\breve{\mathfrak{A}} * \breve{\mathfrak{C}}) \cap_P (\breve{\mathfrak{B}} * \breve{\mathfrak{C}})$.

XIV. $(\breve{\mathfrak{A}} \cup_P \breve{\mathfrak{B}}) * \breve{\mathfrak{C}} = (\breve{\mathfrak{A}} * \breve{\mathfrak{C}}) \cup_P (\breve{\mathfrak{B}} * \breve{\mathfrak{C}})$.

XV. $(\breve{\mathfrak{A}} \cap_P \breve{\mathfrak{B}}) \# \breve{\mathfrak{C}} = (\breve{\mathfrak{A}} \# \breve{\mathfrak{C}}) \cap_P (\breve{\mathfrak{B}} \# \breve{\mathfrak{C}})$.

XVI. $(\breve{\mathfrak{A}} \cup_P \breve{\mathfrak{B}}) \# \breve{\mathfrak{C}} = (\breve{\mathfrak{A}} \# \breve{\mathfrak{C}}) \cup_P (\breve{\mathfrak{B}} \# \breve{\mathfrak{C}})$.

XVII. $(\breve{\mathfrak{A}} @ \breve{\mathfrak{B}}) @ \breve{\mathfrak{C}} \neq \breve{\mathfrak{A}} @ (\breve{\mathfrak{B}} @ \breve{\mathfrak{C}})$.

XVIII. $(\breve{\mathfrak{A}} \$ \breve{\mathfrak{B}}) \$ \breve{\mathfrak{C}} \neq \breve{\mathfrak{A}} \$ (\breve{\mathfrak{B}} \$ \breve{\mathfrak{C}})$.

XIX. $(\breve{\mathfrak{A}} * \breve{\mathfrak{B}}) * \breve{\mathfrak{C}} \neq \breve{\mathfrak{A}} * (\breve{\mathfrak{B}} * \breve{\mathfrak{C}})$.

XX. $(\breve{\mathfrak{A}} \# \breve{\mathfrak{B}}) \# \breve{\mathfrak{C}} \neq \breve{\mathfrak{A}} \# (\breve{\mathfrak{B}} \# \breve{\mathfrak{C}})$.

Proof: the proofs (i)-(xvi) are obvious.

XXI.

L.H.S $(\breve{\mathfrak{A}} @ \breve{\mathfrak{B}}) =$

$$\left\{\left(\breve{\mathfrak{s}}, \begin{array}{c}\left\langle\left[\frac{\eth^-_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) + \eth^-_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}})}{2}, \frac{\eth^+_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) + \eth^+_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}})}{2}\right], \left[\frac{\xi^-_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) + \xi^-_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}})}{2}, \frac{\xi^+_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) + \xi^+_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}})}{2}\right]\right\rangle, \\ \left\langle\frac{\eth_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) + \eth_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}})}{2}, \frac{\xi_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) + \xi_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}})}{2}\right\rangle\end{array}\right) \middle| \breve{\mathfrak{s}} \in \mathfrak{S}\right\}.$$

$(\breve{\mathfrak{A}} @ \breve{\mathfrak{B}}) @ \breve{\mathfrak{C}} =$

$$\left\{\left(\breve{\mathfrak{s}}, \begin{array}{c}\left\langle\begin{array}{c}\left[\frac{1}{2}\left(\frac{\eth^-_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) + \eth^-_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}}) + 2\eth^-_{\breve{\mathfrak{C}}}(\breve{\mathfrak{s}})}{2}, \frac{\eth^+_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) + \eth^+_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}}) + 2\eth^+_{\breve{\mathfrak{C}}}(\breve{\mathfrak{s}})}{2}\right)\right], \\ \left[\frac{1}{2}\left(\frac{\xi^-_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) + \xi^-_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}}) + 2\xi^-_{\breve{\mathfrak{C}}}(\breve{\mathfrak{s}})}{2}, \frac{\xi^+_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) + \xi^+_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}}) + 2\xi^+_{\breve{\mathfrak{C}}}(\breve{\mathfrak{s}})}{2}\right)\right]\end{array}\right\rangle, \\ \left\langle\frac{1}{2}\left(\frac{\eth_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) + \eth_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}}) + 2\eth_{\breve{\mathfrak{C}}}(\breve{\mathfrak{s}})}{2}, \frac{\xi_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) + \xi_{\breve{\mathfrak{B}}}(\breve{\mathfrak{s}}) + 2\xi_{\breve{\mathfrak{C}}}(\breve{\mathfrak{s}})}{2}\right)\right\rangle\end{array}\right) \middle| \breve{\mathfrak{s}} \in \mathfrak{S}\right\}.$$

R.H.S $(\breve{\mathfrak{B}} @ \breve{\mathfrak{C}}) =$

$$\left\{\left(\tilde{\mathfrak{s}}, \begin{matrix}\left\langle\left[\frac{ð_{\breve{\mathfrak{B}}}^{-}(\tilde{\mathfrak{s}}) + ð_{\breve{\mathfrak{C}}}^{-}(\tilde{\mathfrak{s}})}{2}, \frac{ð_{\breve{\mathfrak{B}}}^{+}(\tilde{\mathfrak{s}}) + ð_{\breve{\mathfrak{C}}}^{+}(\tilde{\mathfrak{s}})}{2}\right], \left[\frac{ξ_{\breve{\mathfrak{B}}}^{-}(\tilde{\mathfrak{s}}) + ξ_{\breve{\mathfrak{C}}}^{-}(\tilde{\mathfrak{s}})}{2}, \frac{ξ_{\breve{\mathfrak{B}}}^{+}(\tilde{\mathfrak{s}}) + ξ_{\breve{\mathfrak{C}}}^{+}(\tilde{\mathfrak{s}})}{2}\right]\right\rangle, \\ \left\langle\frac{ð_{\breve{\mathfrak{B}}}(\tilde{\mathfrak{s}}) + ð_{\breve{\mathfrak{C}}}(\tilde{\mathfrak{s}})}{2}, \frac{ξ_{\breve{\mathfrak{B}}}(\tilde{\mathfrak{s}}) + ξ_{\breve{\mathfrak{C}}}(\tilde{\mathfrak{s}})}{2}\right\rangle\end{matrix}\right) \middle| \tilde{\mathfrak{s}} \in \mathfrak{S}\right\}.$$

$$\breve{\mathfrak{A}} @ (\breve{\mathfrak{B}} @ \breve{\mathfrak{C}}) =$$

$$\left\{\left(\tilde{\mathfrak{s}}, \begin{matrix}\left\langle\begin{matrix}\left[\frac{1}{2}\left(\frac{2ð_{\breve{\mathfrak{A}}}^{-}(\tilde{\mathfrak{s}}) + ð_{\breve{\mathfrak{B}}}^{-}(\tilde{\mathfrak{s}}) + ð_{\breve{\mathfrak{C}}}^{-}(\tilde{\mathfrak{s}})}{2}, \frac{2ð_{\breve{\mathfrak{A}}}^{+}(\tilde{\mathfrak{s}}) + ð_{\breve{\mathfrak{B}}}^{+}(\tilde{\mathfrak{s}}) + ð_{\breve{\mathfrak{C}}}^{+}(\tilde{\mathfrak{s}})}{2}\right)\right], \\ \left[\frac{1}{2}\left(\frac{2ξ_{\breve{\mathfrak{A}}}^{-}(\tilde{\mathfrak{s}}) + ξ_{\breve{\mathfrak{B}}}^{-}(\tilde{\mathfrak{s}}) + ξ_{\breve{\mathfrak{C}}}^{-}(\tilde{\mathfrak{s}})}{2}, \frac{2ξ_{\breve{\mathfrak{A}}}^{+}(\tilde{\mathfrak{s}}) + ξ_{\breve{\mathfrak{B}}}^{+}(\tilde{\mathfrak{s}}) + ξ_{\breve{\mathfrak{C}}}^{+}(\mathfrak{s})}{2}\right)\right]\end{matrix}\right\rangle, \\ \left\langle\frac{1}{2}\left(\frac{2ð_{\breve{\mathfrak{A}}}(\tilde{\mathfrak{s}}) + ð_{\breve{\mathfrak{B}}}(\tilde{\mathfrak{s}}) + ð_{\breve{\mathfrak{C}}}(\tilde{\mathfrak{s}})}{2}, \frac{2ξ_{\breve{\mathfrak{A}}}(\tilde{\mathfrak{s}}) + ξ_{\breve{\mathfrak{B}}}(\tilde{\mathfrak{s}}) + ξ_{\breve{\mathfrak{C}}}(\tilde{\mathfrak{s}})}{2}\right)\right\rangle\end{matrix}\right) \middle| \tilde{\mathfrak{s}} \in \mathfrak{S}\right\}.$$

Hence

$$(\breve{\mathfrak{A}} @ \breve{\mathfrak{B}}) @ \breve{\mathfrak{C}} \neq \breve{\mathfrak{A}} @ (\breve{\mathfrak{B}} @ \breve{\mathfrak{C}}).$$

Similarly, equations can also be proved.

Example 2. Consider the US $\mathfrak{S} = \{\tilde{\mathfrak{s}}_1, \tilde{\mathfrak{s}}_2, \tilde{\mathfrak{s}}_3, \tilde{\mathfrak{s}}_4, \tilde{\mathfrak{s}}_5\}$. Let $\breve{\mathfrak{A}}$, $\breve{\mathfrak{B}}$ and $\breve{\mathfrak{C}}$ are CIFSs of $\mathfrak{S}$ given by

$\breve{\mathfrak{A}} = \{(\langle[0.1,0.5], [0.1,0.4]\rangle, \langle 0.5,0.1\rangle), (\langle[0.4,0.5], [0.2,0.3]\rangle, \langle 0.3, 0.2\rangle),$

$(\langle[0.4,0.5], [0.1,0.2]\rangle, \langle 0.5,0.3\rangle), (\langle[0.2,0.4], [0.2,0.2]\rangle, \langle 0.1, 0.7\rangle), (\langle[0.4,0.5], [0.4,0.4]\rangle, \langle 0.2, 0.3\rangle)\}.$

$\breve{\mathfrak{B}} = \{(\langle[0.2,0.5], [0.4,0.5]\rangle, \langle 0.5, 0.5\rangle), (\langle[0.3,0.4], [0.3,0.4]\rangle, \langle 0.6,0.1\rangle),$

$(\langle[0.1,0.5], [0.2,0.3]\rangle, \langle 0.5,0.1\rangle), (\langle[0.4,0.5], [0.1,0.4]\rangle, \langle 0.4,0.2\rangle), (\langle[0.1,0.3], [0.2,0.5]\rangle, \langle 0.4,0.5\rangle)\}.$

$\breve{\mathfrak{C}} = \{(\langle[0.2,0.4], [0.4,0.6]\rangle, \langle 0.7, 0.3\rangle), (\langle[0.1,0.5], [0.3,0.5]\rangle, \langle 0.4,0.6\rangle),$

$(\langle[0.3,0.5], [0.4,0.4]\rangle, \langle 0.5,0.1\rangle), (\langle[0.4,0.6], [0.2,0.4]\rangle, \langle 0.4,0.3\rangle), (\langle[0.2,0.6], [0.1,0.3]\rangle, \langle 0.8,0.2\rangle)\}.$

Then

I.

$$(\breve{\mathfrak{A}} @ \breve{\mathfrak{B}}) @ \breve{\mathfrak{C}} \neq \breve{\mathfrak{A}} @ (\breve{\mathfrak{B}} @ \breve{\mathfrak{C}}).$$

L.H.S $(\breve{\mathfrak{A}} @ \breve{\mathfrak{B}}) = \{(\langle[0.15,0.50], [0.25,0.45]\rangle, \langle 0.50,0.30\rangle),$

$(\langle[0.35,0.45], [0.25,0.35]\rangle, \langle 0.45, 0.15\rangle), (\langle[0.25,0.50], [0.15,0.25]\rangle, \langle 0.50,0.20\rangle),$

$(\langle[0.30,0.45], [0.15,0.30]\rangle, \langle 0.25, 0.45\rangle), (\langle[0.25,0.40], [0.30,0.45]\rangle, \langle 0.30, 0.40\rangle)\}.$

$(\breve{\mathfrak{A}} @ \breve{\mathfrak{B}}) @ \breve{\mathfrak{C}}$
$= \{(\langle[0.18,0.45], [0.33,0.53]\rangle, \langle 0.60,0.30\rangle), (\langle[0.23,0.48], [0.28,0.43]\rangle, \langle 0.43, 0.36\rangle),$

$(\langle[0.28,0.50], [0.28,0.33]\rangle, \langle 0.50,0.15\rangle), (\langle[0.35,0.53], [0.18,0.35]\rangle, \langle 0.33, 0.38\rangle),$

$(\langle[0.23,0.50], [0.20,0.38]\rangle, \langle 0.55, 0.30\rangle)\}.$

R.H.S $(\breve{\mathfrak{B}} @ \breve{\mathfrak{C}}) = \{(\langle[0.20,0.45], [0.40,0.55]\rangle, \langle 0.60,0.40\rangle)$

$(\langle[0.20,0.45], [0.30,0.45]\rangle, \langle 0.50, 0.35\rangle), (\langle[0.20,0.50], [0.30,0.35]\rangle, \langle 0.50,0.10\rangle),$

$(\langle[0.40,0.55],[0.15,0.40]\rangle,\langle 0.40,0.25\rangle),(\langle[0.15,0.45],[0.15,0.40]\rangle,\langle 0.60,0.35\rangle)\}.$

$\breve{\mathfrak{A}} @ (\breve{\mathfrak{B}} @ \breve{\mathfrak{C}})$
$=\{(\langle[0.15,0.48],[0.25,0.48]\rangle,\langle 0.55,0.25\rangle),(\langle[0.30,0.48],[0.25,0.38]\rangle,\langle 0.40,0.28\rangle),$

$(\langle[0.30,0.50],[0.20,0.28]\rangle,\langle 0.50,0.20\rangle),(\langle[0.30,0.48],[0.18,0.30]\rangle,\langle 0.25,0.48\rangle),$

$(\langle[0.28,0.48],[0.28,0.40]\rangle,\langle 0.40,0.33\rangle)\}.$

Hence

$(\breve{\mathfrak{A}} @ \breve{\mathfrak{B}}) @ \breve{\mathfrak{C}} \neq \breve{\mathfrak{A}} @ (\breve{\mathfrak{B}} @ \breve{\mathfrak{C}}).$

II.

$(\breve{\mathfrak{A}} \$ \breve{\mathfrak{B}}) \$ \breve{\mathfrak{C}} \neq \breve{\mathfrak{A}} \$ (\breve{\mathfrak{B}} \$ \breve{\mathfrak{C}}).$

L.H.S $(\breve{\mathfrak{A}} \$ \breve{\mathfrak{B}})=\{(\langle[0.14,0.50],[0.20,0.45]\rangle,\langle 0.50,0.22\rangle),$

$(\langle[0.35,0.45],[0.24,0.35]\rangle,\langle 0.42,0.14\rangle),(\langle[0.20,0.50],[0.14,0.24]\rangle,\langle 0.50,0.17\rangle),$

$(\langle[0.28,0.45],[0.14,0.28]\rangle,\langle 0.20,0.37\rangle),(\langle[0.20,0.39],[0.28,0.45]\rangle,\langle 0.28,0.39\rangle)\}.$

$(\breve{\mathfrak{A}} \$ \breve{\mathfrak{B}}) \$ \breve{\mathfrak{C}}$
$=\{(\langle[0.17,0.45],[0.28,0.52]\rangle,\langle 0.59,0.26\rangle),(\langle[0.19,0.47],[0.27,0.42]\rangle,\langle 0.41,0.29\rangle),$

$(\langle[0.24,0.50],[0.24,0.31]\rangle,\langle 0.50,0.13\rangle),(\langle[0.34,0.52],[0.17,0.34]\rangle,\langle 0.28,0.34\rangle),$

$(\langle[0.20,0.48],[0.17,0.37]\rangle,\langle 0.48,0.28\rangle)\}.$

R.H.S $(\breve{\mathfrak{B}} \$ \breve{\mathfrak{C}})=\{(\langle[0.20,0.45],[0.40,0.550]\rangle,\langle 0.59,0.39\rangle),$

$(\langle[0.17,0.45],[0.30,0.45]\rangle,\langle 0.49,0.24\rangle),(\langle[0.17,0.50],[0.28,0.35]\rangle,\langle 0.50,0.10\rangle),$

$(\langle[0.40,0.55],[0.14,0.40]\rangle,\langle 0.40,0.24\rangle),(\langle[0.14,0.42],[0.14,0.39]\rangle,\langle 0.57,0.32\rangle)\}.$

$\breve{\mathfrak{A}} \$ (\breve{\mathfrak{B}} \$ \breve{\mathfrak{C}})$
$=\{(\langle[0.14,0.47],[0.20,0.47]\rangle,\langle 0.54,0.20\rangle),(\langle[0.26,0.47],[0.24,0.37]\rangle,\langle 0.38,0.22\rangle),$

$(\langle[0.26,0.50],[0.17,0.26]\rangle,\langle 0.50,0.17\rangle),(\langle[0.28,0.47],[0.17,0.28]\rangle,\langle 0.20,0.41\rangle),$

$(\langle[0.24,0.46],[0.24,0.39]\rangle,\langle 0.34,0.31\rangle)\}.$

Hence

$(\breve{\mathfrak{A}} \$ \breve{\mathfrak{B}}) \$ \breve{\mathfrak{C}} \neq \breve{\mathfrak{A}} \$ (\breve{\mathfrak{B}} \$ \breve{\mathfrak{C}}).$

III.

$(\breve{\mathfrak{A}} * \breve{\mathfrak{B}}) * \breve{\mathfrak{C}} \neq \breve{\mathfrak{A}} * (\breve{\mathfrak{B}} * \breve{\mathfrak{C}}).$

L.H.S $(\breve{\mathfrak{A}} * \breve{\mathfrak{B}})=\{(\langle[0.15,0.40],[0.24,0.38]\rangle,\langle 0.40,0.29\rangle),$

$(\langle[0.31,0.38],[0.24,0.31]\rangle,\langle 0.38,0.15\rangle),(\langle[0.24,0.40],[0.15,0.24]\rangle,\langle 0.40,0.19\rangle),$

$(\langle[0.28,0.38],[0.15,0.28]\rangle,\langle 0.24,0.39\rangle),(\langle[0.24,0.35],[0.28,0.38]\rangle,\langle 0.28,0.35\rangle)\}.$

$(\breve{\mathfrak{A}} * \breve{\mathfrak{B}}) * \breve{\mathfrak{C}}$
$=\{(\langle[0.17,0.34],[0.29,0.40]\rangle,\langle 0.43,0.27\rangle),(\langle[0.20,0.37],[0.25,0.35]\rangle,\langle 0.34,0.34\rangle),$

$(\langle[0.25,0.38],[0.26,0.29]\rangle,\langle 0.38,0.14\rangle),(\langle[0.31,0.40],[0.17,0.31]\rangle,\langle 0.29,0.31\rangle),$

$(\langle[0.21,0.39],[0.18,0.30]\rangle,\langle 0.44,0.26\rangle)\}$.

R.H.S $(\breve{\mathfrak{B}} * \breve{\mathfrak{C}}) = \{(\langle[0.02,0.08],[0.07,0.12]\rangle,\langle 0.13,0.07\rangle),$

$(\langle[0.01,0.08],[0.04,0.08]\rangle,\langle 0.10,0.03\rangle),(\langle[0.01,0.10],[0.04,0.05]\rangle,\langle 0.10,0.00\rangle),$

$(\langle[0.07,0.12],[0.01,0.07]\rangle,\langle 0.07,0.03\rangle),(\langle[0.01,0.08],[0.01,0.07]\rangle,\langle 0.12,0.05\rangle)\}$.

$\breve{\mathfrak{A}} * (\breve{\mathfrak{B}} * \breve{\mathfrak{C}})$
$= \{(\langle[0.00,0.02],[0.00,0.02]\rangle,\langle 0.03,0.00\rangle),(\langle[0.00,0.02],[0.00,0.01]\rangle,\langle 0.01,0.00\rangle),$

$(\langle[0.00,0.02],[0.00,0.01]\rangle,\langle 0.02,0.00\rangle),(\langle[0.01,0.02],[0.00,0.01]\rangle,\langle 0.00,0.01\rangle),$

$(\langle[0.00,0.02],[0.00,0.01]\rangle,\langle 0.01,0.01\rangle)\}$.

Hence

$$(\breve{\mathfrak{A}} * \breve{\mathfrak{B}}) * \breve{\mathfrak{C}} \neq \breve{\mathfrak{A}} * (\breve{\mathfrak{B}} * \breve{\mathfrak{C}}).$$

IV.

$$(\breve{\mathfrak{A}} \# \breve{\mathfrak{B}}) \# \breve{\mathfrak{C}} \neq \breve{\mathfrak{A}} \# (\breve{\mathfrak{B}} \# \breve{\mathfrak{C}}).$$

L.H.S $(\breve{\mathfrak{A}} \# \breve{\mathfrak{B}}) = \{(\langle[0.13,0.50],[0.16,0.44]\rangle,\langle 0.50,0.17\rangle),$

$(\langle[0.34,0.44],[0.24,0.34]\rangle,\langle 0.40,0.13\rangle),(\langle[0.16,0.50],[0.13,0.24]\rangle,\langle 0.50,0.15\rangle),$

$(\langle[0.27,0.44],[0.13,0.27]\rangle,\langle 0.16,0.31\rangle),(\langle[0.16,0.38],[0.27,0.44]\rangle,\langle 0.27,0.38\rangle)\}$.

$(\breve{\mathfrak{A}} \# \breve{\mathfrak{B}}) \# \breve{\mathfrak{C}}$
$= \{(\langle[0.16,0.44],[0.23,0.51]\rangle,\langle 0.58,0.21\rangle),(\langle[0.15,0.47],[0.27,0.41]\rangle,\langle 0.40,0.22\rangle),$

$(\langle[0.21,0.50],[0.20,0.30]\rangle,\langle 0.50,0.12\rangle),(\langle[0.32,0.51],[0.16,0.32]\rangle,\langle 0.23,0.31\rangle),$

$(\langle[0.18,0.46],[0.15,0.36]\rangle,\langle 0.40,0.26\rangle)\}$.

R.H.S $(\breve{\mathfrak{B}} \# \breve{\mathfrak{C}}) = \{(\langle[0.20,0.44],[0.40,0.55]\rangle,\langle 0.58,0.38\rangle),$

$(\langle[0.15,0.44],[0.30,0.44]\rangle,\langle 0.48,0.17\rangle)\}(\langle[0.15,0.50],[0.27,0.34]\rangle,\langle 0.50,0.10\rangle),$

$(\langle[0.40,0.55],[0.13,0.40]\rangle,\langle 0.40,0.24\rangle),(\langle[0.13,0.40],[0.13,0.38]\rangle,\langle 0.53,0.29\rangle)$.

$\breve{\mathfrak{A}} \# (\breve{\mathfrak{B}} \# \breve{\mathfrak{C}})$
$= \{(\langle[0.13,0.47],[0.16,0.46]\rangle,\langle 0.54,0.16\rangle),(\langle[0.22,0.47],[0.24,0.36]\rangle,\langle 0.37,0.18\rangle),$

$(\langle[0.22,0.50],[0.15,0.25]\rangle,\langle 0.50,0.15\rangle),(\langle[0.27,0.46],[0.16,0.27]\rangle,\langle 0.16,0.36\rangle),$

$(\langle[0.20,0.44],[0.20,0.39]\rangle,\langle 0.29,0.29\rangle)\}$.

Hence

$$(\breve{\mathfrak{A}} \# \breve{\mathfrak{B}}) * \breve{\mathfrak{C}} \neq \breve{\mathfrak{A}} \# (\breve{\mathfrak{B}} \# \breve{\mathfrak{C}}).$$

Definition 9. Let $\mathfrak{S}$ be a non-empty set. Let

$$\breve{\mathfrak{A}} = \{(\breve{\mathfrak{s}},\langle[\eth^-_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}),\eth^+_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}})],[\xi^-_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}),\xi^+_{\breve{\mathfrak{A}}}(\mathfrak{s})]\rangle,\langle\eth_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}),\xi_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}})\rangle) \mid \breve{\mathfrak{s}} \in \mathfrak{S}\},$$

be a CIFS in $\mathfrak{S}$.

Then, the necessity OP of CIFS under P - order is defined as follows:

$$\Box_P\breve{\mathfrak{A}} = \{(\breve{\mathfrak{s}}, \langle[\eth^-_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}), \eth^+_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}})], [\xi^-_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}), 1-\eth^+_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}})]\rangle, \langle\eth_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}), 1-\eth_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}})\rangle) \mid \breve{\mathfrak{s}} \in \mathfrak{S}\},$$

and the possibility OP of CIFS under P - order is defined as follows:

$$\Diamond_P\breve{\mathfrak{A}} = \{(\breve{\mathfrak{s}}, \langle[\eth^-_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}), 1-\xi^+_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}})], [\xi^-_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}), \xi^+_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}})]\rangle, \langle 1-\xi_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}), \xi_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}})\rangle) \mid \breve{\mathfrak{s}} \in \mathfrak{S}\}.$$

Example 3. Consider the US $\mathfrak{S} = \{\breve{\mathfrak{s}}_1, \breve{\mathfrak{s}}_2, \breve{\mathfrak{s}}_3, \breve{\mathfrak{s}}_4, \breve{\mathfrak{s}}_5\}$. Let $\breve{\mathfrak{A}}$ and $\breve{\mathfrak{B}}$ be two CIFSs of $\mathfrak{S}$ given by

$\breve{\mathfrak{A}} = \{(\langle[0.1,0.5],[0.1,0.4]\rangle,\langle 0.5,0.1\rangle),(\langle[0.4,0.5],[0.2,0.3]\rangle,\langle 0.3,0.2\rangle),$

$(\langle[0.4,0.5],[0.1,0.2]\rangle,\langle 0.5,0.3\rangle),(\langle[0.2,0.4],[0.2,0.2]\rangle,\langle 0.1,0.7\rangle),(\langle[0.4,0.5],[0.4,0.4]\rangle,\langle 0.2,0.3\rangle)\}.$

$\breve{\mathfrak{B}} = \{(\langle[0.2,0.5],[0.4,0.5]\rangle,\langle 0.5,0.5\rangle),(\langle[0.3,0.4],[0.3,0.4]\rangle,\langle 0.6,0.1\rangle),$

$(\langle[0.1,0.5],[0.2,0.3]\rangle,\langle 0.5,0.1\rangle),(\langle[0.4,0.5],[0.1,0.4]\rangle,\langle 0.4,0.2\rangle),(\langle[0.1,0.3],[0.2,0.5]\rangle,\langle 0.4,0.5\rangle)\}.$

Then, the necessity and possibility of OPs on CIFSs $\breve{\mathfrak{A}}$ and $\breve{\mathfrak{B}}$ are given below.

Table 2. The necessity and possibility of OPs on CIFSs.

	$\breve{\mathfrak{s}}_1$	$\breve{\mathfrak{s}}_2$	$\breve{\mathfrak{s}}_3$	$\breve{\mathfrak{s}}_4$	$\breve{\mathfrak{s}}_5$
$\Box_P\breve{\mathfrak{A}}$	$\langle[0.1,0.5],[0.1,0.5]\rangle$, $\langle 0.5,0.5\rangle$	$\langle[0.4,0.5],[0.2,0.5]\rangle$, $\langle 0.3,0.7\rangle$	$\langle[0.4,0.5],[0.1,0.5]\rangle$, $\langle 0.5,0.5\rangle$	$\langle[0.2,0.4],[0.2,0.6]\rangle$, $\langle 0.1,0.9\rangle$	$\langle[0.4,0.5],[0.4,0.5]\rangle$, $\langle 0.2,0.8\rangle$
$\Box_P\breve{\mathfrak{B}}$	$\langle[0.2,0.5],[0.4,0.5]\rangle$, $\langle 0.5,0.5\rangle$	$\langle[0.3,0.4],[0.3,0.6]\rangle$, $\langle 0.6,0.4\rangle$	$\langle[0.1,0.5],[0.2,0.5]\rangle$, $\langle 0.5,0.5\rangle$	$\langle[0.4,0.5],[0.1,0.5]\rangle$, $\langle 0.4,0.6\rangle$	$\langle[0.1,0.3],[0.2,0.7]\rangle$, $\langle 0.4,0.6\rangle$
$\Diamond_P\breve{\mathfrak{A}}$	$\langle[0.1,0.6],[0.1,0.4]\rangle$, $\langle 0.9,0.1\rangle$	$\langle[0.4,0.7],[0.2,0.3]\rangle$, $\langle 0.8,0.2\rangle$	$\langle[0.4,0.8],[0.1,0.2]\rangle$, $\langle 0.7,0.3\rangle$	$\langle[0.2,0.8],[0.2,0.2]\rangle$, $\langle 0.3,0.7\rangle$	$\langle[0.4,0.6],[0.4,0.4]\rangle$, $\langle 0.7,0.3\rangle$
$\Diamond_P\breve{\mathfrak{B}}$	$\langle[0.2,0.5],[0.4,0.5]\rangle$, $\langle 0.5,0.5\rangle$	$\langle[0.3,0.6],[0.3,0.4]\rangle$, $\langle 0.9,0.1\rangle$	$\langle[0.1,0.7],[0.2,0.3]\rangle$, $\langle 0.9,0.1\rangle$	$\langle[0.4,0.6],[0.1,0.4]\rangle$, $\langle 0.8,0.2\rangle$	$\langle[0.1,0.5],[0.2,0.5]\rangle$, $\langle 0.5,0.5\rangle$

It is to be noted that in *Example 3*, $\Box_P\breve{\mathfrak{A}}$, $\Diamond_P\breve{\mathfrak{A}}$, $\Box_P\breve{\mathfrak{B}}$ and $\Diamond_P\breve{\mathfrak{B}}$ are CIFSs.

Definition 10. For any CIFS

$$\breve{\mathfrak{A}} = \{(\langle[\eth^-_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}), \eth^+_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}})], [\xi^-_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}), \xi^+_{\breve{\mathfrak{A}}}(\mathfrak{s})]\rangle, \langle\eth_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}), \xi_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}})\rangle) \mid \breve{\mathfrak{s}} \in \mathfrak{S}\},$$

of $\mathfrak{S}$ and $\acute{\alpha} \in [0,1]$ the OP $\mathfrak{D}_{\acute{\alpha}}(\breve{\mathfrak{A}})$ is defined as

$$\mathfrak{D}_{\acute{\alpha}}(\breve{\mathfrak{A}}) =$$

$$\left\{\left(\begin{array}{c}\breve{\mathfrak{s}}, \left\langle\left[\eth^-_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}), \eth^+_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) + \acute{\alpha}\left(1-\eth^+_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) - \xi^+_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}})\right)\right], \left[\xi^-_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}), \xi^+_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) + (1-\acute{\alpha})\left(1-\eth^+_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) - \xi^+_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}})\right)\right]\right\rangle, \\ \left\langle\eth_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) + \acute{\alpha}\left(1-\eth_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) - \xi_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}})\right), \xi_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) + (1-\acute{\alpha})\left(1-\eth_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) - \xi_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}})\right)\right\rangle\end{array}\right) \middle| \breve{\mathfrak{s}} \in \mathfrak{S}\right\},$$

is called $\acute{\alpha}-\mathrm{modal\ OP}$ of CIFS.

Remark 1. For all CIFS $\breve{\mathfrak{A}}$ and for all $\acute{\alpha} \in [0,1]$:

I. $\mathfrak{D}_{\acute{\alpha}}(\breve{\mathfrak{A}}) = \Box_P\breve{\mathfrak{A}}$ when $\acute{\alpha} = 0$.

II. $\mathfrak{D}_{\acute{\alpha}}(\breve{\mathfrak{A}}) = \Diamond_P\breve{\mathfrak{A}}$ when $\acute{\alpha} = 1$.

Proof: let $\breve{\mathfrak{A}} = \{(\langle[\eth^-_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}), \eth^+_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}})], [\xi^-_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}), \xi^+_{\breve{\mathfrak{A}}}(\mathfrak{s})]\rangle, \langle\eth_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}), \xi_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}})\rangle) \mid \breve{\mathfrak{s}} \in \mathfrak{S}\}$.

I.

$$\mathfrak{D}_{\acute{\alpha}}(\breve{\mathfrak{A}}) =$$

$$\left\{\left(\begin{array}{c}\widetilde{\mathfrak{s}}, \left\langle\left[\eth^{-}_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}), \eth^{+}_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}) + \acute{\alpha}\left(1 - \eth^{+}_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}) - \xi^{+}_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}})\right)\right], \left[\xi^{-}_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}), \xi^{+}_{\breve{\mathfrak{A}}}(\mathfrak{s}) + (1-\acute{\alpha})\left(1 - \eth^{+}_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}) - \xi^{+}_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}})\right)\right]\right\rangle, \\ \left\langle \eth_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}) + \acute{\alpha}\left(1 - \eth_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}) - \xi_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}})\right), \xi_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}) + (1-\acute{\alpha})\left(1 - \eth_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}) - \xi_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}})\right)\right\rangle\end{array}\right) \middle| \widetilde{\mathfrak{s}} \in \mathfrak{S}\right\}.$$

When

$\acute{\alpha} = 0$

$= \{(\mathfrak{s}, \langle[\eth^{-}_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}), \eth^{+}_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}})], [\xi^{-}_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}), 1 - \eth^{+}_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}})]\rangle, \langle \eth_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}), 1 - \eth_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}})\rangle) | \widetilde{\mathfrak{s}} \in \mathfrak{S}\}$

$= \square_{P}\breve{\mathfrak{A}}.$

Hence

$\mathfrak{D}_{\acute{\alpha}}(\breve{\mathfrak{A}}) = \square_{P}\breve{\mathfrak{A}}$ when $\acute{\alpha} = 0$.

Similarly, II can also be proved.

Definition 11. For any CIFS,

$\breve{\mathfrak{A}} = \{(\langle[\eth^{-}_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}), \eth^{+}_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}})], [\xi^{-}_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}), \xi^{+}_{\breve{\mathfrak{A}}}(\mathfrak{s})]\rangle, \langle \eth_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}), \xi_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}})\rangle) | \widetilde{\mathfrak{s}} \in \mathfrak{S}\}$,

of $\mathfrak{S}$ and for any $\acute{\alpha}, \acute{\beta} \in [0,1] \ni \acute{\alpha} + \acute{\beta} \leq 1$ the $(\acute{\alpha}, \acute{\beta})-$ modal OP $\mathfrak{F}_{\acute{\alpha},\acute{\beta}}(\breve{\mathfrak{A}})$ of CIFS is defined as

$\mathfrak{F}_{\acute{\alpha},\acute{\beta}}(\breve{\mathfrak{A}}) =$

$$\left\{\left(\begin{array}{c}\widetilde{\mathfrak{s}}, \left\langle\left[\eth^{-}_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}), \eth^{+}_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}) + \acute{\alpha}\left(1 - \eth^{+}_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}) - \xi^{+}_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}})\right)\right], \left[\xi^{-}_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}), \xi^{+}_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}) + \acute{\beta}\left(1 - \eth^{+}_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}) - \xi^{+}_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}})\right)\right]\right\rangle, \\ \left\langle \eth_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}) + \acute{\alpha}\left(1 - \eth_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}) - \xi_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}})\right), \xi_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}) + \acute{\beta}\left(1 - \eth_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}) - \xi_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}})\right)\right\rangle\end{array}\right) \middle| \widetilde{\mathfrak{s}} \in \mathfrak{S}\right\}.$$

Remark 2. For all CIFS $\breve{\mathfrak{A}}$ and for all $\acute{\alpha}, \acute{\beta} \in [0,1] \ni 0 \leq \acute{\alpha} + \acute{\beta} \leq 1$:

I. $\mathfrak{F}_{\acute{\alpha},\acute{\beta}}(\breve{\mathfrak{A}}) = \square_{P}\breve{\mathfrak{A}}$ when $\acute{\alpha} = 0$ and $\acute{\beta} = 1$.

II. $\mathfrak{F}_{\acute{\alpha},\acute{\beta}}(\breve{\mathfrak{A}}) = \diamond_{P}\breve{\mathfrak{A}}$ when $\acute{\alpha} = 1$ and $\acute{\beta} = 0$.

Proof: let $\breve{\mathfrak{A}} = \{(\langle[\eth^{-}_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}), \eth^{+}_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}})], [\xi^{-}_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}), \xi^{+}_{\breve{\mathfrak{A}}}(\mathfrak{s})]\rangle, \langle \eth_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}), \xi_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}})\rangle) | \widetilde{\mathfrak{s}} \in \mathfrak{S}\}$.

I.

$\mathfrak{F}_{\acute{\alpha},\acute{\beta}}(\breve{\mathfrak{A}}) =$

$$\left\{\left(\begin{array}{c}\widetilde{\mathfrak{s}}, \left\langle\left[\eth^{-}_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}), \eth^{+}_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}) + \acute{\alpha}\left(1 - \eth^{+}_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}) - \xi^{+}_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}})\right)\right], \left[\xi^{-}_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}), \xi^{+}_{\breve{\mathfrak{A}}}(\mathfrak{s}) + \acute{\beta}\left(1 - \eth^{+}_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}) - \xi^{+}_{\breve{\mathfrak{A}}}(\mathfrak{s})\right)\right]\right\rangle, \\ \left\langle \eth_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}) + \acute{\alpha}\left(1 - \eth_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}) - \xi_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}})\right), \xi_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}) + \acute{\beta}\left(1 - \eth_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}) - \xi_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}})\right)\right\rangle\end{array}\right) \middle| \widetilde{\mathfrak{s}} \in \mathfrak{S}\right\}.$$

When

$\acute{\alpha} = 0$ and $\acute{\beta} = 1$

$= \{(\widetilde{\mathfrak{s}}, \langle[\eth^{-}_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}), \eth^{+}_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}})], [\xi^{-}_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}), 1 - \eth^{+}_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}})]\rangle, \langle \eth_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}}), 1 - \eth_{\breve{\mathfrak{A}}}(\widetilde{\mathfrak{s}})\rangle) | \widetilde{\mathfrak{s}} \in \mathfrak{S}\}$

$= \square_{P}\breve{\mathfrak{A}}.$

Hence

$\mathfrak{F}_{\acute{\alpha},\acute{\beta}}(\breve{\mathfrak{A}}) = \square_{P}\breve{\mathfrak{A}}$ when $\acute{\alpha} = 0$ and $\acute{\beta} = 1$.

Similarly, II can also be proved.

Definition 12. Let $\mathfrak{S}$ be a non-empty finite set, and let $\acute{\alpha}, \acute{\beta} \in [0,1]$. Then, for any CIFS $\breve{\mathfrak{A}}$.

I.

$$\mathfrak{G}_{\acute{\alpha},\acute{\beta}}(\breve{\mathfrak{A}}) = \left\{\left(\breve{\mathfrak{s}}, \left\langle\left[\acute{\alpha}\eth^{-}_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}), \acute{\alpha}\eth^{+}_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}})\right], \left[\acute{\beta}\xi^{-}_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}), \acute{\beta}\xi^{+}_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}})\right]\right\rangle, \left\langle\acute{\alpha}\eth_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}), \acute{\beta}\xi_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}})\right\rangle\right) \middle| \breve{\mathfrak{s}} \in \mathfrak{S}\right\}.$$

II.

$$\mathfrak{H}_{\acute{\alpha},\acute{\beta}}(\breve{\mathfrak{A}}) = \left\{\left(\begin{array}{c}\breve{\mathfrak{s}}, \left\langle\left[\acute{\alpha}\eth^{-}_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}), \acute{\alpha}\eth^{+}_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}})\right], \left[\xi^{-}_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}), \xi^{+}_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) + \acute{\beta}\left(1 - \eth^{+}_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) - \xi^{+}_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}})\right)\right]\right\rangle, \\ \left\langle\acute{\alpha}\eth_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}), \xi_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) + \acute{\beta}\left(1 - \eth_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) - \xi_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}})\right)\right\rangle\end{array}\right) \middle| \breve{\mathfrak{s}} \in \mathfrak{S}\right\}.$$

III.

$$\mathfrak{H}^{*}_{\acute{\alpha},\acute{\beta}}(\breve{\mathfrak{A}}) = \left\{\left(\begin{array}{c}\breve{\mathfrak{s}}, \left\langle\left[\acute{\alpha}\eth^{-}_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}), \acute{\alpha}\eth^{+}_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}})\right], \left[\xi^{-}_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}), \xi^{+}_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) + \acute{\beta}\left(1 - \acute{\alpha}\eth^{+}_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) - \xi^{+}_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}})\right)\right]\right\rangle, \\ \left\langle\acute{\alpha}\eth_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}), \xi_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) + \acute{\beta}\left(1 - \acute{\alpha}\eth_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) - \xi_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}})\right)\right\rangle\end{array}\right) \middle| \breve{\mathfrak{s}} \in \mathfrak{S}\right\}.$$

IV.

$$\mathfrak{I}_{\acute{\alpha},\acute{\beta}}(\breve{\mathfrak{A}}) = \left\{\left(\begin{array}{c}\breve{\mathfrak{s}}, \left\langle\left[\eth^{-}_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}), \eth^{+}_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) + \acute{\alpha}\left(1 - \eth^{+}_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) - \xi^{+}_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}})\right)\right], \left[\acute{\beta}\xi^{-}_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}), \acute{\beta}\xi^{+}_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}})\right]\right\rangle, \\ \left\langle\eth_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) + \acute{\alpha}\left(1 - \eth_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) - \xi_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}})\right), \acute{\beta}\xi_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}})\right\rangle\end{array}\right) \middle| \breve{\mathfrak{s}} \in \mathfrak{S}\right\}.$$

V.

$$\mathfrak{I}^{*}_{\acute{\alpha},\acute{\beta}}(\breve{\mathfrak{A}}) = \left\{\left(\begin{array}{c}\breve{\mathfrak{s}}, \left\langle\left[\eth^{-}_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}), \eth^{+}_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) + \acute{\alpha}\left(1 - \eth^{+}_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) - \acute{\beta}\xi^{+}_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}})\right)\right], \left[\acute{\beta}\xi^{-}_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}), \acute{\beta}\xi^{+}_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}})\right]\right\rangle, \\ \left\langle\eth_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) + \acute{\alpha}\left(1 - \eth_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}}) - \acute{\beta}\xi_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}})\right), \acute{\beta}\xi_{\breve{\mathfrak{A}}}(\breve{\mathfrak{s}})\right\rangle\end{array}\right) \middle| \breve{\mathfrak{s}} \in \mathfrak{S}\right\}.$$

Proposition 2. For every CIFS $\breve{\mathfrak{A}}$ and $\breve{\mathfrak{B}}$ for every $\acute{\alpha}, \acute{\beta}, \acute{\gamma} \in [0,1]$.

I. $\mathfrak{G}_{\acute{\alpha},\acute{\beta}}(\breve{\mathfrak{A}} \cap_{P} \breve{\mathfrak{B}}) = \mathfrak{G}_{\acute{\alpha},\acute{\beta}}(\breve{\mathfrak{A}}) \cap_{P} \mathfrak{G}_{\acute{\alpha},\acute{\beta}}(\breve{\mathfrak{B}})$.

II. $\mathfrak{G}_{\acute{\alpha},\acute{\beta}}(\breve{\mathfrak{A}} \cup_{P} \breve{\mathfrak{B}}) = \mathfrak{G}_{\acute{\alpha},\acute{\beta}}(\breve{\mathfrak{A}}) \cup_{P} \mathfrak{G}_{\acute{\alpha},\acute{\beta}}(\breve{\mathfrak{B}})$.

III. $\acute{\alpha} \leq \acute{\gamma} \Longrightarrow \mathfrak{H}_{\acute{\alpha},\acute{\beta}}(\breve{\mathfrak{A}}) \subseteq_{P} \mathfrak{H}_{\acute{\gamma},\acute{\beta}}(\breve{\mathfrak{A}})$.

IV. $\acute{\gamma} \leq \acute{\beta} \Longrightarrow \mathfrak{H}_{\acute{\alpha},\acute{\beta}}(\breve{\mathfrak{A}}) \subseteq_{P} \mathfrak{H}_{\acute{\alpha},\acute{\gamma}}(\breve{\mathfrak{A}})$.

V. $\mathfrak{H}^{*}_{\acute{\alpha},\acute{\beta}}(\breve{\mathfrak{A}}) = \overline{\Diamond_{P}\overline{\overline{\breve{\mathfrak{A}}}}}$ when $\acute{\alpha} = 1$ and $\acute{\beta} = 1$.

VI. $\mathfrak{H}^{*}_{\acute{\alpha},\acute{\beta}}(\breve{\mathfrak{A}}) = \breve{\mathfrak{A}}$ when $\acute{\alpha} = 1$ and $\acute{\beta} = 0$.

VII. $\acute{\alpha} \leq \acute{\gamma} \Longrightarrow \mathfrak{I}_{\acute{\alpha},\acute{\beta}}(\breve{\mathfrak{A}}) \subseteq_{P} \mathfrak{I}_{\acute{\gamma},\acute{\beta}}(\breve{\mathfrak{A}})$.

VIII. $\acute{\gamma} \leq \acute{\beta} \Rightarrow \mathfrak{J}_{\acute{\alpha},\acute{\beta}}(\breve{\mathfrak{A}}) \subseteq_P \mathfrak{J}_{\acute{\alpha},\acute{\gamma}}(\breve{\mathfrak{A}})$.

IX. $\mathfrak{J}^{*}{}_{\acute{\alpha},\acute{\beta}}(\breve{\mathfrak{A}}) = \overline{\square_P \overline{\overline{\breve{\mathfrak{A}}}}}$ when $\acute{\alpha} = 1$ and $\acute{\beta} = 1$.

X. $\mathfrak{J}^{*}{}_{\acute{\alpha},\acute{\beta}}(\breve{\mathfrak{A}}) = \breve{\mathfrak{A}}$ when $\acute{\alpha} = 0$ and $\acute{\beta} = 1$.

Proof: let $\breve{\mathfrak{A}} = \{(\langle[\eth^{-}_{\breve{\mathfrak{A}}}(\widehat{\mathfrak{s}}), \eth^{+}_{\breve{\mathfrak{A}}}(\widehat{\mathfrak{s}})], [\xi^{-}_{\breve{\mathfrak{A}}}(\widehat{\mathfrak{s}}), \xi^{+}_{\breve{\mathfrak{A}}}(\mathfrak{s})]\rangle, \langle\eth_{\breve{\mathfrak{A}}}(\widehat{\mathfrak{s}}), \xi_{\breve{\mathfrak{A}}}(\widehat{\mathfrak{s}})\rangle) \mid \widehat{\mathfrak{s}} \in \mathfrak{S}\}$.

$\breve{\mathfrak{B}} = \{(\langle[\eth^{-}_{\breve{\mathfrak{B}}}(\widehat{\mathfrak{s}}), \eth^{+}_{\breve{\mathfrak{B}}}(\widehat{\mathfrak{s}})], [\xi^{-}_{\breve{\mathfrak{B}}}(\widehat{\mathfrak{s}}), \xi^{+}_{\breve{\mathfrak{B}}}(\widehat{\mathfrak{s}})]\rangle, \langle\eth_{\breve{\mathfrak{B}}}(\widehat{\mathfrak{s}}), \xi_{\breve{\mathfrak{B}}}(\widehat{\mathfrak{s}})\rangle) \mid \widehat{\mathfrak{s}} \in \mathfrak{S}\}$.

I.

L.H.S$\mathfrak{G}_{\acute{\alpha},\acute{\beta}}(\breve{\mathfrak{A}} \cap_P \breve{\mathfrak{B}}) =$

$$\left\{\left(\widehat{\mathfrak{s}}, \left\langle \begin{array}{c} \left[\min\left(\acute{\alpha}\eth^{-}_{\breve{\mathfrak{A}}}(\widehat{\mathfrak{s}}), \acute{\alpha}\eth^{-}_{\breve{\mathfrak{B}}}(\widehat{\mathfrak{s}})\right), \min\left(\acute{\alpha}\eth^{+}_{\breve{\mathfrak{A}}}(\widehat{\mathfrak{s}}), \acute{\alpha}\eth^{+}_{\breve{\mathfrak{B}}}(\widehat{\mathfrak{s}})\right)\right], \\ \left[\max\left(\acute{\beta}\xi^{-}_{\breve{\mathfrak{A}}}(\widehat{\mathfrak{s}}), \acute{\beta}\xi^{-}_{\breve{\mathfrak{B}}}(\widehat{\mathfrak{s}})\right), \max\left(\acute{\beta}\xi^{+}_{\breve{\mathfrak{A}}}(\widehat{\mathfrak{s}}), \acute{\beta}\xi^{-}_{\breve{\mathfrak{B}}}(\widehat{\mathfrak{s}})\right)\right] \end{array} \right\rangle, \atop \left\langle \min\left(\acute{\alpha}\eth_{\breve{\mathfrak{A}}}(\widehat{\mathfrak{s}}), \acute{\alpha}\eth_{\breve{\mathfrak{B}}}(\widehat{\mathfrak{s}})\right), \max\left(\acute{\beta}\xi_{\breve{\mathfrak{A}}}(\widehat{\mathfrak{s}}), \acute{\beta}\xi_{\breve{\mathfrak{B}}}(\widehat{\mathfrak{s}})\right)\right\rangle \right) \middle| \widehat{\mathfrak{s}} \in \mathfrak{S}\right\}.$$

II.

R.H.S $\mathfrak{G}_{\acute{\alpha},\acute{\beta}}(\breve{\mathfrak{A}}) \cap_P \mathfrak{G}_{\acute{\alpha},\acute{\beta}}(\breve{\mathfrak{B}}) =$

$$\left\{\left(\widehat{\mathfrak{s}}, \left\langle \begin{array}{c} \left[\min\left(\acute{\alpha}\eth^{-}_{\breve{\mathfrak{A}}}(\widehat{\mathfrak{s}}), \acute{\alpha}\eth^{-}_{\breve{\mathfrak{B}}}(\widehat{\mathfrak{s}})\right), \min\left(\acute{\alpha}\eth^{+}_{\breve{\mathfrak{A}}}(\widehat{\mathfrak{s}}), \acute{\alpha}\eth^{+}_{\breve{\mathfrak{B}}}(\widehat{\mathfrak{s}})\right)\right], \\ \left[\max\left(\acute{\beta}\xi^{-}_{\breve{\mathfrak{A}}}(\widehat{\mathfrak{s}}), \acute{\beta}\xi^{-}_{\breve{\mathfrak{B}}}(\widehat{\mathfrak{s}})\right), \max\left(\acute{\beta}\xi^{+}_{\breve{\mathfrak{A}}}(\widehat{\mathfrak{s}}), \acute{\beta}\xi^{-}_{\breve{\mathfrak{B}}}(\widehat{\mathfrak{s}})\right)\right] \end{array} \right\rangle, \atop \left\langle \min\left(\acute{\alpha}\eth_{\breve{\mathfrak{A}}}(\widehat{\mathfrak{s}}), \acute{\alpha}\eth_{\breve{\mathfrak{B}}}(\widehat{\mathfrak{s}})\right), \max\left(\acute{\beta}\xi_{\breve{\mathfrak{A}}}(\widehat{\mathfrak{s}}), \acute{\beta}\xi_{\breve{\mathfrak{B}}}(\widehat{\mathfrak{s}})\right)\right\rangle \right) \middle| \widehat{\mathfrak{s}} \in \mathfrak{S}\right\}.$$

Hence

$$\mathfrak{G}_{\acute{\alpha},\acute{\beta}}(\breve{\mathfrak{A}} \cap_P \breve{\mathfrak{B}}) = \mathfrak{G}_{\acute{\alpha},\acute{\beta}}(\breve{\mathfrak{A}}) \cap_P \mathfrak{G}_{\acute{\alpha},\acute{\beta}}(\breve{\mathfrak{B}}).$$

Similarly, II can also be proved.

III.

$$\mathfrak{H}_{\acute{\alpha},\acute{\beta}}(\breve{\mathfrak{A}}) = \left\{\left(\begin{array}{c} \widehat{\mathfrak{s}}, \left\langle\left[\acute{\alpha}\eth^{-}_{\breve{\mathfrak{A}}}(\widehat{\mathfrak{s}}), \acute{\alpha}\eth^{+}_{\breve{\mathfrak{A}}}(\widehat{\mathfrak{s}})\right], \left[\xi^{-}_{\breve{\mathfrak{A}}}(\widehat{\mathfrak{s}}), \xi^{+}_{\breve{\mathfrak{A}}}(\widehat{\mathfrak{s}}) + \acute{\beta}\left(1 - \eth^{+}_{\breve{\mathfrak{A}}}(\widehat{\mathfrak{s}}) - \xi^{+}_{\breve{\mathfrak{A}}}(\widehat{\mathfrak{s}})\right)\right]\right\rangle, \\ \left\langle\acute{\alpha}\eth_{\breve{\mathfrak{A}}}(\widehat{\mathfrak{s}}), \xi_{\breve{\mathfrak{A}}}(\widehat{\mathfrak{s}}) + \acute{\beta}\left(1 - \eth_{\breve{\mathfrak{A}}}(\widehat{\mathfrak{s}}) - \xi_{\breve{\mathfrak{A}}}(\widehat{\mathfrak{s}})\right)\right\rangle \end{array}\right) \middle| \widehat{\mathfrak{s}} \in \mathfrak{S}\right\},$$

$$\mathfrak{H}_{\acute{\gamma},\acute{\beta}}(\breve{\mathfrak{A}}) = \left\{\left(\begin{array}{c} \widehat{\mathfrak{s}}, \left\langle\left[\acute{\gamma}\eth^{-}_{\breve{\mathfrak{A}}}(\widehat{\mathfrak{s}}), \acute{\gamma}\eth^{+}_{\breve{\mathfrak{A}}}(\widehat{\mathfrak{s}})\right], \left[\xi^{-}_{\breve{\mathfrak{A}}}(\widehat{\mathfrak{s}}), \xi^{+}_{\breve{\mathfrak{A}}}(\widehat{\mathfrak{s}}) + \acute{\beta}\left(1 - \eth^{+}_{\breve{\mathfrak{A}}}(\widehat{\mathfrak{s}}) - \xi^{+}_{\breve{\mathfrak{A}}}(\widehat{\mathfrak{s}})\right)\right]\right\rangle, \\ \left\langle\acute{\gamma}\eth_{\breve{\mathfrak{A}}}(\widehat{\mathfrak{s}}), \xi_{\breve{\mathfrak{A}}}(\widehat{\mathfrak{s}}) + \acute{\beta}\left(1 - \eth_{\breve{\mathfrak{A}}}(\widehat{\mathfrak{s}}) - \xi_{\breve{\mathfrak{A}}}(\widehat{\mathfrak{s}})\right)\right\rangle \end{array}\right) \middle| \widehat{\mathfrak{s}} \in \mathfrak{S}\right\},$$

$\acute{\alpha} \leq \acute{\gamma}$,

$\acute{\alpha}\eth^{-}_{\breve{\mathfrak{A}}}(\widehat{\mathfrak{s}}) \subseteq \acute{\gamma}\eth^{-}_{\breve{\mathfrak{A}}}(\widehat{\mathfrak{s}})$ and $\acute{\alpha}\eth^{+}_{\breve{\mathfrak{A}}}(\widehat{\mathfrak{s}}) \subseteq \acute{\gamma}\eth^{+}_{\breve{\mathfrak{A}}}(\widehat{\mathfrak{s}})$.

Similarly, $\acute{\alpha}\eth_{\breve{\mathfrak{A}}}(\widehat{\mathfrak{s}}) \subseteq \acute{\gamma}\eth_{\breve{\mathfrak{A}}}(\widehat{\mathfrak{s}})$.

Hence $\acute{\alpha} \leq \acute{\gamma} \Rightarrow \mathfrak{H}_{\acute{\alpha},\acute{\beta}}(\breve{\mathfrak{A}}) \subseteq_P \mathfrak{H}_{\acute{\gamma},\acute{\beta}}(\breve{\mathfrak{A}})$.

Similarly, I to IV can also be proved.

4 | Decision–making Approach Using Proposed Cubic Intuitionistic Fuzzy Operations under P-order

We have discussed and illustrated a decision-making strategy in this section.

The set of $\dot{m}$ alternatives, denoted as $\breve{\mathfrak{A}} = \{\breve{\mathfrak{A}}_1, \breve{\mathfrak{A}}_2, \ldots\ldots, \breve{\mathfrak{A}}_{\dot{m}}\}$ is evaluated based on the set of $\dot{n}$ criteria, represented by $\breve{\mathfrak{C}} = \{\breve{\mathfrak{C}}_1, \breve{\mathfrak{C}}_2, \ldots\ldots, \breve{\mathfrak{C}}_{\dot{n}}\}$. The preferences for these alternatives are expressed as CIFNs

$$\breve{\mathfrak{A}}_{i,j} = \left(\langle[\eth_{i,j}^-, \eth_{i,j}^+], [\xi_{i,j}^-, \xi_{i,j}^+]\rangle, \langle\eth_{i,j}, \xi_{i,j}\rangle\right), \; i = 1, 2, \ldots\ldots, \dot{m}, \; j = 1,2, \ldots\ldots\ldots, \dot{n}.$$

Now, let us discuss the following steps, which clearly explain the new approach.

Step 1. (Structure of the CIFN decision-making matrix).

Compile all relevant data for each alternative in the context of CIFNs. As a result, we can develop a decision-making matrix $\mathfrak{U} = (\mathfrak{u})_{\dot{m}\times\dot{n}}$ as,

$$\mathfrak{U} = \begin{pmatrix} \mathfrak{u}_{11} & \cdots & \mathfrak{u}_{1\dot{n}} \\ \cdots & \cdots & \cdots \\ \mathfrak{u}_{\dot{m}1} & \cdots & \mathfrak{u}_{\dot{m}\dot{n}} \end{pmatrix}.$$

Step 2. (Normalize the decision-making matrix).

Normalization is not necessary if all attributes are of the same type. However, if there are multiple types of criteria, such as profit and cost, we can use the following normalized formula to transform cost type criterion into profit type:

$$\mathfrak{V}_{i,j} = \begin{cases} \left(\langle[\eth_{i,j}^-, \eth_{i,j}^+], [\xi_{i,j}^-, \xi_{i,j}^+]\rangle, \langle\eth_{i,j}, \xi_{i,j}\rangle\right), & \text{for benefit type criterion,} \\ \left(\langle[\xi_{i,j}^-, \xi_{i,j}^+], [\eth_{i,j}^-, \eth_{i,j}^+]\rangle, \langle\xi_{i,j}, \eth_{i,j}\rangle\right), & \text{for cost type criterion.} \end{cases} \tag{1}$$

Thus, the normalized CIFN decision matrix, $\mathfrak{V} = (\mathfrak{v})_{\dot{m}\times\dot{n}}$ is obtained as

$$\mathfrak{V} = \begin{pmatrix} \mathfrak{v}_{11} & \cdots & \mathfrak{v}_{1\dot{n}} \\ \cdots & \cdots & \cdots \\ \mathfrak{v}_{1} & \cdots & \mathfrak{v}_{\dot{m}\dot{n}} \end{pmatrix}.$$

Step 3. (OP computation on CIFN).

Utilizing *Definitions 10-12*, calculate the proposed OPs on CIFNs for each alternative.

Step 4. (Calculation of arithmetic mean).

To use *Definition 5* to calculate the arithemetic mean value.

Step 5. (Ranking alternatives).

The score function of CIFN under P-order [80] is described as

$$\mathcal{S}(\breve{\mathfrak{A}}_{i,j}) = \frac{\eth_{i,j}^- + \eth_{i,j}^+ - \xi_{i,j}^- - \xi_{i,j}^+}{2} + \eth_{i,j} - \xi_{i,j},$$

where $-2 \leq \mathcal{S}(\breve{\mathfrak{A}}_{i,j}) \leq 2$.

Determine the score value and select the best option accordingly.

Example 4. This illustration is taken from Garg and Kaur [80].

An Operating System (OS) assigns resources to various jobs through the job-scheduling process. The order, priority, and CPU allocation of the job in the job queue are set by the job schedular. This ensures that no

work is derived from CPU allocation and that all jobs are finished fairly on schedule time. Most OSs, such as UNIX, Windows, iOS, Android, etc., include built-in job-scheduling capabilities. Additionally, certain job-scheduling features are also included in the OSs, including Database Management System (DBMS), Enterprise Resource Planning (ERP), and Business Process Management (BPM).

Suppose a programmer needs to schedule jobs $\breve{\mathfrak{A}}_1, \breve{\mathfrak{A}}_2, \breve{\mathfrak{A}}_3$ and $\breve{\mathfrak{A}}_4$ according to the criteria $\breve{\mathfrak{C}}_1$, $\breve{\mathfrak{C}}_2$, and $\breve{\mathfrak{C}}_3$, where

$\breve{\mathfrak{C}}_1$: automated restart in case of failure.

$\breve{\mathfrak{C}}_2$: number of parallel jobs permitted for a user.

$\breve{\mathfrak{C}}_3$: execution time assigned to a user.

The programmer aims to schedule the jobs efficiently to minimize execution time and prevent extended waiting periods for CPU allocation.

The computation steps of the suggested method are as follows:

Step 1. The decision-maker uses CIFNs to evaluate these options, taking into account the three broad attributes previously discussed. The CIF decision matrix is presented in *Table 3*.

Table 3. The alternatives in terms of CIFNs.

	$\breve{\mathfrak{C}}_1$	$\breve{\mathfrak{C}}_2$	$\breve{\mathfrak{C}}_3$
$\breve{\mathfrak{A}}_1$	$\langle[0.50, 0.80], [0.10, 0.20]\rangle$, $\langle 0.40, 0.20\rangle$	$\langle[0.20, 0.30], [0.40, 0.50]\rangle$, $\langle 0.50, 0.20\rangle$	$\langle[0.40, 0.60], [0.20, 0.30]\rangle$, $\langle 0.20, 0.70\rangle$
$\breve{\mathfrak{A}}_2$	$\langle[0.20, 0.30], [0.40, 0.50]\rangle$, $\langle 0.40, 0.60\rangle$	$\langle[0.80, 0.90], [0.01, 0.10]\rangle$, $\langle 0.40, 0.20\rangle$	$\langle[0.20, 0.60], [0.10, 0.20]\rangle$, $\langle 0.40, 0.50\rangle$
$\breve{\mathfrak{A}}_3$	$\langle[0.50, 0.60], [0.20, 0.30]\rangle$, $\langle 0.20, 0.40\rangle$	$\langle[0.40, 0.60], [0.20, 0.30]\rangle$, $\langle 0.30, 0.40\rangle$	$\langle[0.50, 0.70], [0.20, 0.30]\rangle$, $\langle 0.30, 0.50\rangle$
$\breve{\mathfrak{A}}_4$	$\langle[0.30, 0.70], [0.10, 0.30]\rangle$, $\langle 0.10, 0.30\rangle$	$\langle[0.40, 0.90], [0.05, 0.10]\rangle$, $\langle 0.30, 0.40\rangle$	$\langle[0.40, 0.60], [0.20, 0.30]\rangle$, $\langle 0.60, 0.30\rangle$

Note: table values adapted from [80].

Step 2. Since $\breve{\mathfrak{C}}_1$ and $\breve{\mathfrak{C}}_3$ are cost-type criteria, equation can be used for their normalization. As a result, *Table 4* displays the normalized decision matrix.

Table 4. Normalize the decision-making matrix.

	$\breve{\mathfrak{C}}_1$	$\breve{\mathfrak{C}}_2$	$\breve{\mathfrak{C}}_3$
$\breve{\mathfrak{A}}_1$	$\langle[0.10, 0.20], [0.50, 0.80]\rangle$, $\langle 0.20, 0.40\rangle$	$\langle[0.20, 0.30], [0.40, 0.50]\rangle$, $\langle 0.50, 0.20\rangle$	$\langle[0.20, 0.30], [0.40, 0.60]\rangle$, $\langle 0.70, 0.20\rangle$
$\breve{\mathfrak{A}}_2$	$\langle[0.40, 0.50], [0.20, 0.30]\rangle$, $\langle 0.60, 0.40\rangle$	$\langle[0.80, 0.90], [0.01, 0.10]\rangle$, $\langle 0.40, 0.20\rangle$	$\langle[0.10, 0.20], [0.20, 0.60]\rangle$, $\langle 0.50, 0.40\rangle$
$\breve{\mathfrak{A}}_3$	$\langle[0.20, 0.30], [0.50, 0.60]\rangle$, $\langle 0.40, 0.20\rangle$	$\langle[0.40, 0.60], [0.20, 0.30]\rangle$, $\langle 0.30, 0.40\rangle$	$\langle[0.20, 0.30], [0.50, 0.70]\rangle$, $\langle 0.50, 0.30\rangle$
$\breve{\mathfrak{A}}_4$	$\langle[0.10, 0.30], [0.30,0.70]\rangle$, $\langle 0.30, 0.10\rangle$	$\langle[0.40, 0.90], [0.05, 0.10]\rangle$, $\langle 0.30, 0.40\rangle$	$\langle[0.20, 0.30], [0.40, 0.60]\rangle$, $\langle 0.30, 0.60\rangle$

Note: table values adapted from [80].

Step 3. According to the *Definition 8*, $\mathfrak{D}_{\breve{\alpha}}$ of each alternative is obtained in *Table 5*.

Table 5. $\mathfrak{D}_{\acute{\alpha}}$ operator on CIFNs.

			$\breve{\mathfrak{C}}_1$	$\breve{\mathfrak{C}}_2$	$\breve{\mathfrak{C}}_3$
$\mathfrak{D}_{\acute{\alpha}}$	$\acute{\alpha}=0.2$ $\acute{\beta}=0.3$	$\breve{\mathfrak{A}}_1$	$\langle[0.10,0.20],[0.50,0.80]\rangle$, $\langle 0.28,0.72\rangle$	$\langle[0.20,0.34],[0.40,0.66]\rangle$, $\langle 0.56,0.44\rangle$	$\langle[0.20,0.32],[0.40,0.68]\rangle$, $\langle 0.72,0.28\rangle$
		$\breve{\mathfrak{A}}_2$	$\langle[0.40,0.54],[0.20,0.46]\rangle$, $\langle 0.60,0.40\rangle$	$\langle[0.80,0.90],[0.01,0.10]\rangle$, $\langle 0.48,0.52\rangle$	$\langle[0.10,0.24],[0.20,0.76]\rangle$, $\langle 0.52,0.48\rangle$
		$\breve{\mathfrak{A}}_3$	$\langle[0.20,0.32],[0.50,0.68]\rangle$, $\langle 0.48,0.52\rangle$	$\langle[0.40,0.62],[0.20,0.38]\rangle$, $\langle 0.36,0.64\rangle$	$\langle[0.20,0.30],[0.50,0.70]\rangle$, $\langle 0.54,0.46\rangle$
		$\breve{\mathfrak{A}}_4$	$\langle[0.10,0.30],[0.30,0.70]\rangle$, $\langle 0.42,0.58\rangle$	$\langle[0.40,0.90],[0.05,0.10]\rangle$, $\langle 0.36,0.64\rangle$	$\langle[0.20,0.30],[0.50,0.70]\rangle$, $\langle 0.54,0.46\rangle$
	$\acute{\alpha}=0.5$ $\acute{\beta}=0.5$	$\breve{\mathfrak{A}}_1$	$\langle[0.10,0.20],[0.50,0.80]\rangle$, $\langle 0.40,0.60\rangle$	$\langle[0.20,0.40],[0.40,0.60]\rangle$, $\langle 0.65,0.35\rangle$	$\langle[0.20,0.35],[0.40,0.65]\rangle$, $\langle 0.75,0.25\rangle$
		$\breve{\mathfrak{A}}_2$	$\langle[0.40,0.60],[0.20,0.40]\rangle$, $\langle 0.60,0.40\rangle$	$\langle[0.80,0.90],[0.01,0.10]\rangle$, $\langle 0.60,0.40\rangle$	$\langle[0.10,0.30],[0.20,0.70]\rangle$, $\langle 0.55,0.45\rangle$
		$\breve{\mathfrak{A}}_3$	$\langle[0.20,0.35],[0.50,0.65]\rangle$, $\langle 0.60,0.40\rangle$	$\langle[0.40,0.65],[0.20,0.35]\rangle$, $\langle 0.45,0.55\rangle$	$\langle[0.20,0.30],[0.50,0.70]\rangle$, $\langle 0.60,0.40\rangle$
		$\breve{\mathfrak{A}}_4$	$\langle[0.10,0.30],[0.30,0.70]\rangle$, $\langle 0.60,0.40\rangle$	$\langle[0.40,0.90],[0.05,0.10]\rangle$, $\langle 0.45,0.55\rangle$	$\langle[0.20,0.35],[0.40,0.65]\rangle$, $\langle 0.35,0.65\rangle$
	$\acute{\alpha}=0.4$ $\acute{\beta}=0.2$	$\breve{\mathfrak{A}}_1$	$\langle[0.10,0.20],[0.50,0.80]\rangle$, $\langle 0.20,0.40\rangle$	$\langle[0.10,0.20],[0.50,0.80]\rangle$, $\langle 0.20,0.40\rangle$	$\langle[0.10,0.20],[0.50,0.80]\rangle$, $\langle 0.20,0.40\rangle$
		$\breve{\mathfrak{A}}_2$	$\langle[0.10,0.20],[0.50,0.80]\rangle$, $\langle 0.20,0.40\rangle$	$\langle[0.10,0.20],[0.50,0.80]\rangle$, $\langle 0.20,0.40\rangle$	$\langle[0.10,0.20],[0.50,0.80]\rangle$, $\langle 0.20,0.40\rangle$
		$\breve{\mathfrak{A}}_3$	$\langle[0.10,0.20],[0.50,0.80]\rangle$, $\langle 0.20,0.40\rangle$	$\langle[0.10,0.20],[0.50,0.80]\rangle$, $\langle 0.20,0.40\rangle$	$\langle[0.10,0.20],[0.50,0.80]\rangle$, $\langle 0.20,0.40\rangle$
		$\breve{\mathfrak{A}}_4$	$\langle[0.10,0.20],[0.50,0.80]\rangle$, $\langle 0.20,0.40\rangle$	$\langle[0.10,0.20],[0.50,0.80]\rangle$, $\langle 0.20,0.40\rangle$	$\langle[0.10,0.20],[0.50,0.80]\rangle$, $\langle 0.20,0.40\rangle$

Similarly, by using the *Definitions 10 and 12*, the other OPs $\mathfrak{F}_{\acute{\alpha},\acute{\beta}}$, $\mathfrak{G}_{\acute{\alpha},\acute{\beta}}$, $\mathfrak{H}_{\acute{\alpha},\acute{\beta}}$, $\mathfrak{H}^*{}_{\acute{\alpha},\acute{\beta}}$, $\mathfrak{I}_{\acute{\alpha},\acute{\beta}}$ and $\mathfrak{I}^*{}_{\acute{\alpha},\acute{\beta}}$ can be found.

Step 4. The arithmetic mean values for each are calculated in *Table 6*.

Table 6. The arithmetic mean value for CIFNs.

Operator		Alternatives	Arithmetic Mean Value
$\mathfrak{D}_{\acute{\alpha}}$	$\acute{\alpha}=0.2$	$\breve{\mathfrak{A}}_1$	$\langle[0.17,0.29],[0.43,0.71]\rangle,\langle 0.52,0.48\rangle$
	$\acute{\beta}=0.3$	$\breve{\mathfrak{A}}_2$	$\langle[0.43,0.56],[0.14,0.44]\rangle,\langle 0.53,0.47\rangle$
		$\breve{\mathfrak{A}}_3$	$\langle[0.27,0.41],[0.40,0.59]\rangle,\langle 0.46,0.54\rangle$
		$\breve{\mathfrak{A}}_4$	$\langle[0.23,0.51],[0.25,0.49]\rangle,\langle 0.37,0.63\rangle$
	$\acute{\alpha}=0.5$	$\breve{\mathfrak{A}}_1$	$\langle[0.17,0.32],[0.43,0.68]\rangle,\langle 0.60,0.40\rangle$
	$\acute{\beta}=0.5$	$\breve{\mathfrak{A}}_2$	$\langle[0.43,0.60],[0.14,0.40]\rangle,\langle 0.58,0.42\rangle$
		$\breve{\mathfrak{A}}_3$	$\langle[0.27,0.43],[0.40,0.57]\rangle,\langle 0.55,0.45\rangle$
		$\breve{\mathfrak{A}}_4$	$\langle[0.23,0.52],[0.25,0.48]\rangle,\langle 0.47,0.53\rangle$
	$\acute{\alpha}=0.4$	$\breve{\mathfrak{A}}_1$	$\langle[0.17,0.31],[0.43,0.69]\rangle,\langle 0.57,0.43\rangle$
	$\acute{\beta}=0.2$	$\breve{\mathfrak{A}}_2$	$\langle[0.43,0.59],[0.14,0.41]\rangle,\langle 0.57,0.43\rangle$
		$\breve{\mathfrak{A}}_3$	$\langle[0.27,0.43],[0.40,0.57]\rangle,\langle 0.52,0.48\rangle$
		$\breve{\mathfrak{A}}_4$	$\langle[0.23,0.51],[0.25,0.49]\rangle,\langle 0.43,0.57\rangle$

Similarly, the arithmetic mean value for other OPs $\mathfrak{F}_{\acute{\alpha},\acute{\beta}}$, $\mathfrak{G}_{\acute{\alpha},\acute{\beta}}$, $\mathfrak{H}_{\acute{\alpha},\acute{\beta}}$, $\mathfrak{H}^*{}_{\acute{\alpha},\acute{\beta}}$, $\mathfrak{I}_{\acute{\alpha},\acute{\beta}}$ and $\mathfrak{I}^*{}_{\acute{\alpha},\acute{\beta}}$ can be found.

Step 5. Finally, the score values obtained using the score function for CIFNs P-order are listed in *Table 7*.

Table 7. Score value for CIFNs.

		$\mathfrak{D}_{\acute{\alpha}}$	$\mathfrak{F}_{\acute{\alpha},\acute{\beta}}$	$\mathfrak{G}_{\acute{\alpha},\acute{\beta}}$	$\mathfrak{H}_{\acute{\alpha},\acute{\beta}}$	$\mathfrak{H}^*_{\acute{\alpha},\acute{\beta}}$	$\mathfrak{I}_{\acute{\alpha},\acute{\beta}}$	$\mathfrak{I}^*_{\acute{\alpha},\acute{\beta}}$
$\acute{\alpha}=0.2$, $\acute{\beta}=0.3$	$\breve{\mathfrak{A}}_1$	-0.3067	-0.1483	-0.1033	-0.7583	-0.8890	0.5067	0.5883
	$\breve{\mathfrak{A}}_2$	0.2750	0.3917	0.0262	-0.4417	-0.5990	0.8595	0.9295
	$\breve{\mathfrak{A}}_3$	-0.2333	-0.0667	-0.0833	-0.7200	-0.8440	0.5700	0.6493
	$\breve{\mathfrak{A}}_4$	-0.2683	-0.0933	-0.0842	-0.6967	-0.8037	0.5192	0.6032
$\acute{\alpha}=0.5$, $\acute{\beta}=0.5$	$\breve{\mathfrak{A}}_1$	-0.1167	-0.1167	-0.0583	-0.6167	-0.7667	0.4417	0.5875
	$\breve{\mathfrak{A}}_2$	0.4150	0.4150	0.2075	-0.1933	-0.3850	0.8158	0.9408
	$\breve{\mathfrak{A}}_3$	-0.0333	-0.0333	-0.0167	-0.5667	-037167	0.5167	0.6583
	$\breve{\mathfrak{A}}_4$	-0.0583	-0.0583	-0.0292	-0.5667	-0.7042	0.4792	0.6292
$\acute{\alpha}=0.4$, $\acute{\beta}=0.2$	$\breve{\mathfrak{A}}_1$	-0.1800	-0.0533	0.0433	-0.5900	-0.6620	0.6500	0.8367
	$\breve{\mathfrak{A}}_2$	0.3683	0.4617	0.2127	-0.2217	-0.3137	0.9630	0.9230
	$\breve{\mathfrak{A}}_3$	-0.1000	0.0333	0.0600	-0.5400	-0.6120	0.7133	0.8947
	$\breve{\mathfrak{A}}_4$	-0.1283	0.0117	0.0467	-0.5283	-0.5943	0.6617	0.8537

Similarly, the score values for IFNs and IVIFNs are provided in *Table 8* and *Table 9*.

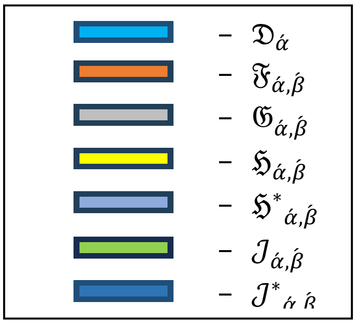
Table 8. Score value for IFNs.

		$\mathfrak{D}_{\acute{\alpha}}$ [9]	$\mathfrak{F}_{\acute{\alpha},\acute{\beta}}$ [9]	$\mathfrak{G}_{\acute{\alpha},\acute{\beta}}$ [9]	$\mathfrak{H}_{\acute{\alpha},\acute{\beta}}$ [9]	$\mathfrak{H}^*_{\acute{\alpha},\acute{\beta}}$ [9]	$\mathfrak{I}_{\acute{\alpha},\acute{\beta}}$ [9]	$\mathfrak{I}^*_{\acute{\alpha},\acute{\beta}}$ [9]
$\acute{\alpha}=0.2$, $\acute{\beta}=0.3$	$\breve{\mathfrak{A}}_1$	0.0667	0.1733	0.0133	-0.2533	-0.3653	0.4400	0.4773
	$\breve{\mathfrak{A}}_2$	0.0400	0.1500	0.0000	-0.2833	-0.4033	0.4333	0.4800
	$\breve{\mathfrak{A}}_3$	-0.0800	0.0700	-0.0100	-0.3100	-0.4060	0.3700	0.4120
	$\breve{\mathfrak{A}}_4$	-0.2667	-0.1000	-0.0500	-0.4067	-0.4787	0.2567	0.3080
$\acute{\alpha}=0.5$, $\acute{\beta}=0.5$	$\breve{\mathfrak{A}}_1$	0.2000	0.2000	0.1000	-0.1667	-0.2833	0.4667	0.5333
	$\breve{\mathfrak{A}}_2$	0.1667	0.1667	0.0833	-0.1667	-0.2917	0.4167	0.5000
	$\breve{\mathfrak{A}}_3$	0.1000	0.1000	0.0500	-0.2500	-0.3500	0.4000	0.4750
	$\breve{\mathfrak{A}}_4$	-0.0667	-0.0667	-0.0333	-0.3833	-0.4583	0.2833	0.3750
$\acute{\alpha}=0.4$, $\acute{\beta}=0.2$	$\breve{\mathfrak{A}}_1$	0.1467	0.2533	0.1333	-0.1333	-0.1893	0.5200	0.6053
	$\breve{\mathfrak{A}}_2$	0.1333	0.2000	0.1333	-0.1667	-0.2267	0.5000	0.6067
	$\breve{\mathfrak{A}}_3$	0.0400	0.1600	0.1000	-0.2000	-0.2480	0.4600	0.5560
	$\breve{\mathfrak{A}}_4$	-0.1333	0.0000	0.04667	-0.3133	-0.3493	0.3600	0.4773

Table 9. Score value for IVIFNs.

		$\mathfrak{D}_{\acute{\alpha}}$ [13]	$\mathfrak{F}_{\acute{\alpha},\acute{\beta}}$ [13]	$\mathfrak{G}_{\acute{\alpha},\acute{\beta}}$ [13]	$\mathfrak{H}_{\acute{\alpha},\acute{\beta}}$ [13]	$\mathfrak{H}^*_{\acute{\alpha},\acute{\beta}}$ [13]	$\mathfrak{I}_{\acute{\alpha},\acute{\beta}}$ [13]	$\mathfrak{I}^*_{\acute{\alpha},\acute{\beta}}$ [13]
$\acute{\alpha}=0.2$, $\acute{\beta}=0.3$	$\breve{\mathfrak{A}}_1$	-0.3467	-0.3217	-0.1167	-0.5050	-0.5370	0.0667	0.1110
	$\breve{\mathfrak{A}}_2$	0.2083	0.2417	0.0262	-0.1583	-0.2223	0.4262	0.4495
	$\breve{\mathfrak{A}}_3$	-0.1533	-0.1367	-0.0733	-0.4100	-0.4580	0.2000	0.2373
	$\breve{\mathfrak{A}}_4$	-0.0017	0.0067	-0.0342	-0.2900	-0.3500	0.2625	0.2952
$\acute{\alpha}=0.5$, $\acute{\beta}=0.5$	$\breve{\mathfrak{A}}_1$	-0.3167	-0.3167	-0.1583	-0.4500	-0.4833	-0.0250	0.0542
	$\breve{\mathfrak{A}}_2$	0.2483	0.2483	0.1242	-0.0267	-0.0933	0.3992	0.4408
	$\breve{\mathfrak{A}}_3$	-0.1333	-0.1333	-0.0667	-0.3167	-0.3667	0.1167	0.1833
	$\breve{\mathfrak{A}}_4$	0.0083	0.0083	0.0042	-0.1833	-0.2458	0.1958	0.2542
$\acute{\alpha}=0.4$, $\acute{\beta}=0.2$	$\breve{\mathfrak{A}}_1$	-0.3267	-0.3067	-0.0200	-0.4567	-0.4727	0.1300	0.2313
	$\breve{\mathfrak{A}}_2$	0.2350	0.2617	0.1463	-0.0550	-0.0870	0.4630	0.5163
	$\breve{\mathfrak{A}}_3$	-0.1400	-0.1267	0.0400	-0.3400	-0.3640	0.2533	0.3387
	$\breve{\mathfrak{A}}_4$	0.0050	0.0117	0.0750	-0.2150	-0.2450	0.3017	0.3763

The following figures represent the score values for each operator for CIFN, IFN, and IVIFN.



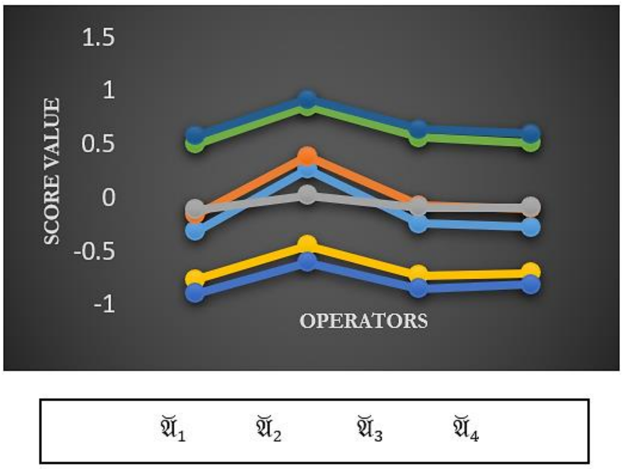


Fig. 1. Score value for CIFN $(\acute{\alpha} = 0.2, \acute{\beta} = 0.3)$.

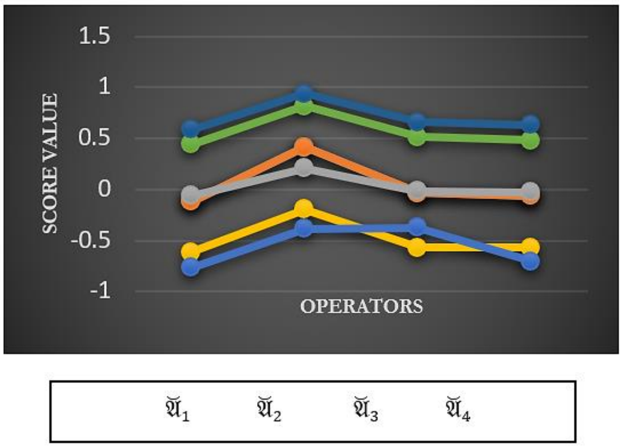


Fig. 2. Score value for CIFN $(\acute{\alpha} = 0.5, \acute{\beta} = 0.5)$.

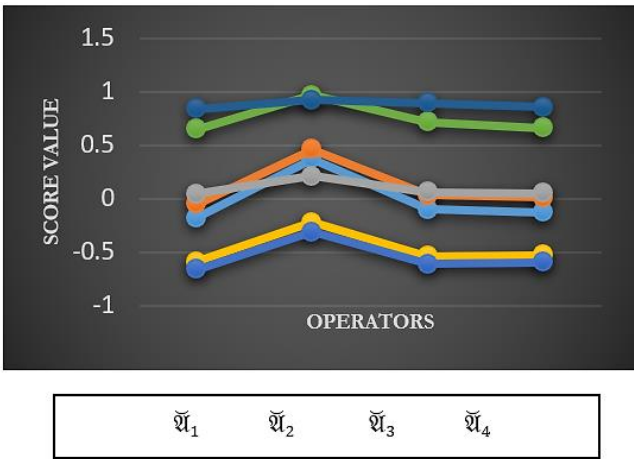


Fig. 3. Score value for CIFN $(\acute{\alpha} = 0.4, \acute{\beta} = 0.2)$.

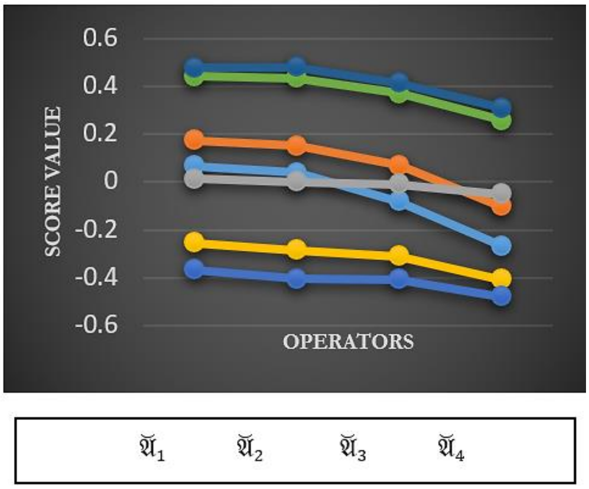


Fig. 4. Score value for IFN $(\acute{\alpha} = 0.2, \acute{\beta} = 0.3)$.

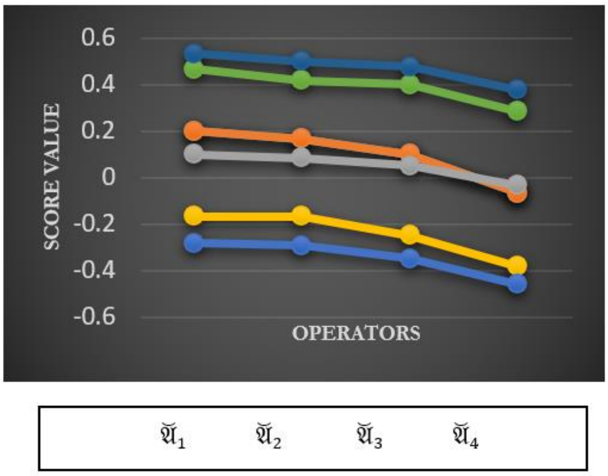


Fig. 5. Score value for IFN $(\acute{\alpha} = 0.5, \acute{\beta} = 0.5)$.

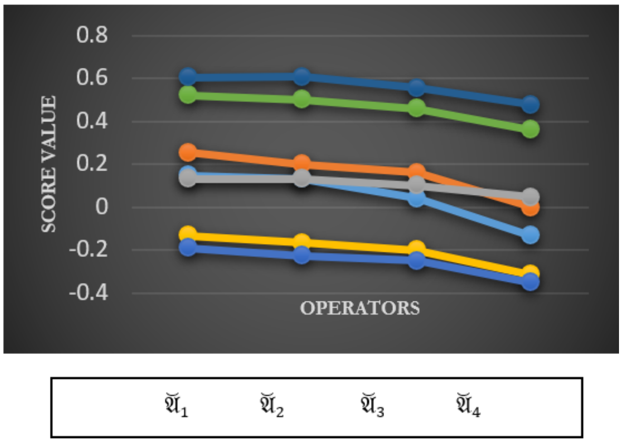


Fig. 6. Score value for IFN $\left(\acute{\alpha} = 0.4, \acute{\beta} = 0.2\right)$.

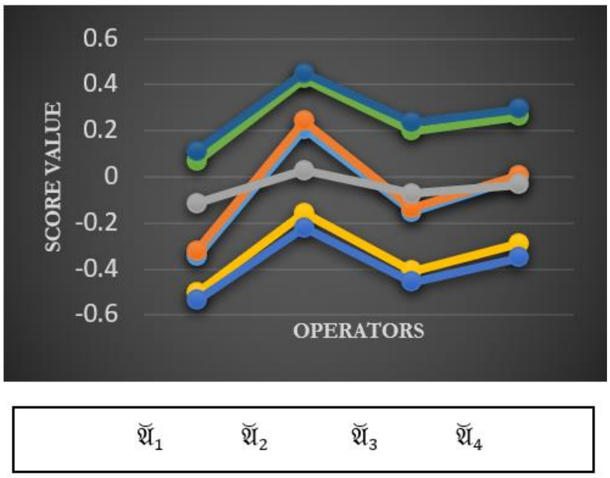


Fig. 7. Score value for IVIFN $\left(\acute{\alpha} = 0.2, \acute{\beta} = 0.3\right)$.

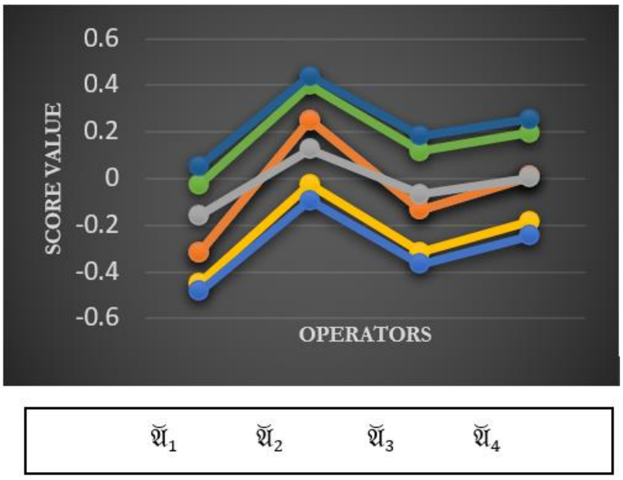


Fig. 8. Score value for IVIFN $\left(\acute{\alpha} = 0.5, \acute{\beta} = 0.5\right)$.

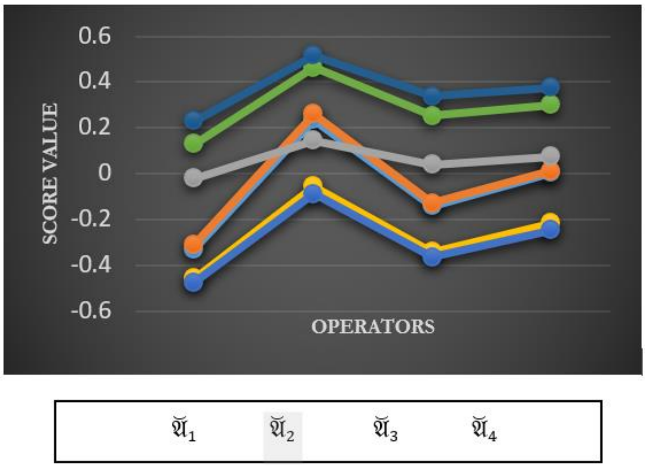


Fig. 9. Score value for IVIFN $\left(\acute{\alpha} = 0.4, \acute{\beta} = 0.2\right)$.

The alternatives for IFNs, IVIFNs, and CIFNs are ranked based on their respective score values, as shown in *Table 10*.

Table 10. Comparative analysis of IFNs, IVIFNs and CIFNs.

Operators	$\acute{\alpha}$ and $\acute{\beta}$ Values	IFNs	IVIFNs	CIFNs
$\mathfrak{D}_{\acute{\alpha}}$	$\acute{\alpha} = 0.2$, $\acute{\beta} = 0.3$	$\breve{\mathfrak{A}}_1 > \breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_4$	$\breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_4 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_1$	$\breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_4 > \breve{\mathfrak{A}}_1$
$\mathfrak{F}_{\acute{\alpha},\acute{\beta}}$		$\breve{\mathfrak{A}}_1 > \breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_4$	$\breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_4 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_1$	$\breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_4 > \breve{\mathfrak{A}}_1$
$\mathfrak{G}_{\acute{\alpha},\acute{\beta}}$		$\breve{\mathfrak{A}}_1 > \breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_4$	$\breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_4 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_1$	$\breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_4 > \breve{\mathfrak{A}}_1$
$\mathfrak{H}_{\acute{\alpha},\acute{\beta}}$		$\breve{\mathfrak{A}}_1 > \breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_4$	$\breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_4 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_1$	$\breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_4 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_1$
$\mathfrak{H}^*{}_{\acute{\alpha},\acute{\beta}}$		$\breve{\mathfrak{A}}_1 > \breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_4$	$\breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_4 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_1$	$\breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_4 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_1$
$\mathfrak{I}_{\acute{\alpha},\acute{\beta}}$		$\breve{\mathfrak{A}}_1 > \breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_4$	$\breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_4 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_1$	$\breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_4 > \breve{\mathfrak{A}}_1$
$\mathfrak{I}^*{}_{\acute{\alpha},\acute{\beta}}$		$\breve{\mathfrak{A}}_1 > \breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_4$	$\breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_4 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_1$	$\breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_4 > \breve{\mathfrak{A}}_1$
$\mathfrak{D}_{\acute{\alpha}}$	$\acute{\alpha} = 0.5$, $\acute{\beta} = 0.5$	$\breve{\mathfrak{A}}_1 > \breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_4$	$\breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_4 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_1$	$\breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_4 > \breve{\mathfrak{A}}_1$
$\mathfrak{F}_{\acute{\alpha},\acute{\beta}}$		$\breve{\mathfrak{A}}_1 > \breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_4$	$\breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_4 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_1$	$\breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_4 > \breve{\mathfrak{A}}_1$
$\mathfrak{G}_{\acute{\alpha},\acute{\beta}}$		$\breve{\mathfrak{A}}_1 > \breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_4$	$\breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_4 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_1$	$\breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_4 > \breve{\mathfrak{A}}_1$
$\mathfrak{H}_{\acute{\alpha},\acute{\beta}}$		$\breve{\mathfrak{A}}_1 = \breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_4$	$\breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_4 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_1$	$\breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_4 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_1$
$\mathfrak{H}^*{}_{\acute{\alpha},\acute{\beta}}$		$\breve{\mathfrak{A}}_1 > \breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_4$	$\breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_4 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_1$	$\breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_4 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_1$
$\mathfrak{I}_{\acute{\alpha},\acute{\beta}}$		$\breve{\mathfrak{A}}_1 > \breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_4$	$\breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_4 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_1$	$\breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_4 > \breve{\mathfrak{A}}_1$
$\mathfrak{I}^*{}_{\acute{\alpha},\acute{\beta}}$		$\breve{\mathfrak{A}}_1 > \breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_4$	$\breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_4 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_1$	$\breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_4 > \breve{\mathfrak{A}}_1$
$\mathfrak{D}_{\acute{\alpha}}$	$\acute{\alpha} = 0.4$, $\acute{\beta} = 0.2$	$\breve{\mathfrak{A}}_1 > \breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_4$	$\breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_4 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_1$	$\breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_4 > \breve{\mathfrak{A}}_1$
$\mathfrak{F}_{\acute{\alpha},\acute{\beta}}$		$\breve{\mathfrak{A}}_1 > \breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_4$	$\breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_4 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_1$	$\breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_4 > \breve{\mathfrak{A}}_1$
$\mathfrak{G}_{\acute{\alpha},\acute{\beta}}$		$\breve{\mathfrak{A}}_1 = \breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_4$	$\breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_4 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_1$	$\breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_4 > \breve{\mathfrak{A}}_1$
$\mathfrak{H}_{\acute{\alpha},\acute{\beta}}$		$\breve{\mathfrak{A}}_1 > \breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_4$	$\breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_4 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_1$	$\breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_4 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_1$
$\mathfrak{H}^*{}_{\acute{\alpha},\acute{\beta}}$		$\breve{\mathfrak{A}}_1 > \breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_4$	$\breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_4 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_1$	$\breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_4 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_1$
$\mathfrak{I}_{\acute{\alpha},\acute{\beta}}$		$\breve{\mathfrak{A}}_1 > \breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_4$	$\breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_4 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_1$	$\breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_4 > \breve{\mathfrak{A}}_1$
$\mathfrak{I}^*{}_{\acute{\alpha},\acute{\beta}}$		$\breve{\mathfrak{A}}_1 > \breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_4$	$\breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_4 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_1$	$\breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_4 > \breve{\mathfrak{A}}_1$

From these results, it is evident that $\breve{\mathfrak{A}}_2$ is consistently the most preferable alternative according to all OPs for IVIFSs and CIFSs across all three cases: $\acute{\alpha} \leq \acute{\beta}, \acute{\alpha} = \acute{\beta}$ and $\acute{\alpha} \geq \acute{\beta}$. We observe that the ranking order coincides with [50]. However, for IFSs, the ranking order is incorrect; $\breve{\mathfrak{A}}_1$ is identified as the best alternative, and in some cases, $\breve{\mathfrak{A}}_1$are $\breve{\mathfrak{A}}_2$ are ranked equally. Thus, IFSs demonstrate inconsistencies. In IVIFSs, all OPs exhibit the same ranking order for all cases $\acute{\alpha} \leq \acute{\beta}, \acute{\alpha} = \acute{\beta}$ and $\acute{\alpha} \geq \acute{\beta}$: $\breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_4 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_1$. However, IVIFSs have only considered the preferences of the alternatives initially. This approach may lose some valuable information that could impact the decision outcomes. Conversely, CIFSs produce two possible ranking orders for all cases $\acute{\alpha} \leq \acute{\beta}, \acute{\alpha} = \acute{\beta}$ and $\acute{\alpha} \geq \acute{\beta}$: $\breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_4 > \breve{\mathfrak{A}}_1$ and $\breve{\mathfrak{A}}_2 > \breve{\mathfrak{A}}_4 > \breve{\mathfrak{A}}_3 > \breve{\mathfrak{A}}_1$.

Hence, we conclude that the proposed decision-making method using a CIF environment is significantly more effective than existing approaches. By simultaneously incorporating both IVIFSs and IFSs, the proposed method provides a more comprehensive evaluation of alternatives.

5 | Conclusion

This article outlines specific operations on CIFSs under P-order. A few of its characteristics are demonstrated and validated using numerical values. Moreover, we applied the work to an MCDM approach. These OPs help manage and process clearly defined data, and they also help handle imprecision and uncertainty by enabling combination, comparison, and modification. Each OP has its own unique advantages depending on the control and requirements of a specific application domain, offering practical and straightforward solutions. Application areas such as pattern recognition, fault diagnosis, reliability optimization of complex systems, machine learning, robotics, economy, control systems, computers, and algebraic structures stand to benefit from these OPs.

The different relations between the above-defined OPs in CIFSs are proved. Publication preparations are now underway for this ongoing study. In our subsequent studies, we plan to employ OPs to assess the effectiveness of our techniques in various image processing contexts, particularly for enhancing images using the intensification OPs. Additionally, the R-order framework still allows for the definition of some OPs across CIFSs.

Acknowledgment

The author will be extremely grateful to the learned reviewers for their valuable comments and suggestions, which helped him a lot in improving the standard of this paper.

Author Contribution

Conceptualization, Saranya M; writing-original & draft preparation and editing, Priyadharshini M; Formal analysis, investigation, and review, Jayanthi D. All authors have read and agreed to the published version of the manuscript.

Funding

The authors declare that no funds, grants, or other support were received during the preparation of this manuscript.

Data Availability

Data presented in this study are available in "Kaur, G., & Garg, H. (2018). Cubic intuitionistic fuzzy aggregation operators. International Journal for Uncertainty Quantification, 8, 405 – 427 [40]. https://doi.org/10.1615/Int.J.UncertaintyQuantification.2018020471

Conflicts of Interest

The authors have no relevant financial or non-financial interests to disclose.

References

[1] Zadeh, L. A. (1965). Fuzzy sets. *Information and control,* 8(3), 338-353. https://doi.org/10.1016/S0019-9958(65)90241-X

[2] Zadeh, L. A. (1975). The concept of a linguistic variable and its application to approximate reasoning-I. *Information sciences*, *8*(3), 199–249. DOI:10.1016/0020-0255(75)90036-5

[3] Atanassov, K. T., & Stoeva, S. (1986). Intuitionistic fuzzy sets. *Fuzzy sets and systems*, *20*(1), 87–96.

[4] Gorzałczany, M. B. (1989). An interval-valued fuzzy inference method—some basic properties. *Fuzzy sets and systems*, *31*(2), 243-251.

[5] Atanassov, K., & Gargov, G. (1989). Interval valued intuitionistic fuzzy sets. *Fuzzy sets and systems*, *31*(3), 343–349. DOI:10.1016/0165-0114(89)90205-4

[6] Atanassov, K. T. (2016). Four new operators over intuitionistic fuzzy sets. *International journal bioautomation*, *20*(S1), S55–S62. https://biomed.bas.bg/bioautomation/2016/vol_20.s1/files/20.s1_07.pdf

[7] Atanassov, K. T. (1989). More on intuitionistic fuzzy sets. *Fuzzy sets and systems*, *33*(1), 37–45. DOI:10.1016/0165-0114(89)90215-7

[8] Atanassov, K. T. (2016). Review and new results on intuitionistic fuzzy sets. *International journal bioautomation*, *20*, S17–S26.

[9] Atanassov, K. T. (1994). New operations defined over the intuitionistic fuzzy sets. *Fuzzy sets and systems*, *61*(2), 137–142. DOI:10.1016/0165-0114(94)90229-1

[10] Atanassov, K. T. (2001). A theorem for basis operators over intuitionistic fuzzy sets. *Mathware & soft computing. 2001 Vol. 8 Núm. 1*, 21-30. https://upcommons.upc.edu/bitstream/handle/2099/3593/vol8-1-Atanasov.pdf

[11] Atanassov, K. T. (2012). *On intuitionistic fuzzy sets theory* (Vol. 283). Springer.

[12] Atanassov, K. T. (2018). Two intuitionistic fuzzy modal-level operators. *Advances in intelligent systems and computing* (Vol. 641, pp. 85–98). Springer Verlag. DOI: 10.1007/978-3-319-66830-7_9

[13] Atanassov, K. T. (1994). Operators over interval valued intuitionistic fuzzy sets. *Fuzzy sets and systems*, *64*(2), 159–174. DOI:10.1016/0165-0114(94)90331-X

[14] Atanassov, K. (2018). On interval-valued intuitionistic fuzzy modal operators. *Notes on intuitionistic fuzzy sets*, *24*(1), 1–12. DOI:10.7546/nifs.2018.24.1.1-12

[15] Rajesh, K., & Srinivasan, R. (2007). Certified Organization) Website: www. *International journal of innovative research in science, engineering and technology (an iso*, *3297*. DOI:10.15680/IJIRSET.2017.0607198

[16] Rajesh, K., & Srinivasan, R. (2017). Modal type operators over interval valued intuitionistic fuzzy sets of second type. *International Journal of fuzzy mathematical archive*, *14*(2), 253-260. DOI:10.22457/ijfma.v14n2a9

[17] Sharma, M., Rajpurohit, B. S., Agnihotri, S., & Rathore, A. K. (2020). Development of fractional order modeling of voltage source converters. *IEEE access*, *8*, 131750-131759.

[18] Citil, M., & Tugrul, F. (2017). Some intuitionistic fuzzy operators. *IFSCOM Mersin, Turkey* (pp. 1-5). https://www.researchgate.net/publication/312529613_Some_intuitionistic_fuzzy_operators

[19] Çitil, M., & Tuğrul, F. (2018). Some new equalities on the intuitionistic fuzzy modal operators. *Sakarya University Journal of Science*, *22*(6), 1524-1531. DOI:10.16984/saufenbilder.338899

[20] Kumar De, S., Biswas, R., & Roy, A. R. (2000). Some operations on intuitionistic fuzzy sets. *Fuzzy sets and systems*, *114*(3), 477–484. DOI:10.1016/s0165-0114(98)00191-2

[21] Ejegwa, P. A., Akowe, S. O., Otene, P. M., & Ikyule, J. M. (2014). An overview on intuitionistic fuzzy sets. *International journal of scientific & technology research*, *3*(3), 142-145.

[22] Eker, E., Tugrul, F., & Citil, M. (2015). New equalities on the intuitionistic fuzzy operators and operations. *Notes on intuitionistic fuzzy sets*, *21*(4), 124-128.

[23] Liu, Q., Ma, C., & Zhou, X. (2008). On properties of some IFS operators and operations. *Notes on intuitionistic fuzzy sets*, *14*(3), 17-24.

[24] Mala, & Shanmugapriya, M. M. (2017). Properties of intuitionistic fuzzy set operators. *International journal of development research, 7*, 9309. https://journalijdr.com/properties-intuitionistic-fuzzy-set-operators

[25] Bhattacharya, J. (2016). A few more on intuitionistic fuzzy set. *Journal of fuzzy set valued analysis*, *2016*(3), 214–222. DOI:10.5899/2016/jfsva-00322

[26] Bhattacharya, J. (2021). Some Results on Certain Properties of Intuitionistic Fuzzy Sets. *Journal of fuzzy extension and applications*, *2*(4), 377–387. DOI:10.22105/jfea.2021.290202.1155

[27] Bhattacharya, J. (2021). Some Special Operations and Related Results on Intuitionistic Fuzzy Sets. *International journal of scientific research in mathematical and statistical sciences*, *8*(4), 10–13. DOI:10.26438/ijsrmss/v8i4.1013

[28] Iqbal, M. N. (2019). Some topological and modal operators over the intuitionistic fuzzy sets of cube root type. *International journal of recent technology and engineering*, *8*(3), 1247–1251. DOI:10.35940/ijrte.C4401.098319

[29] Jun, Y. B., Hur, K., Lee, J. G., & Kim, J. (2021). Crossing cubic structures as an extension of bipolar fuzzy sets. *Annals of fuzzy mathematics and informatics*, *22*(1), 1-15.

[30] Wang, W., Liu, X., & Qin, Y. (2012). Interval-valued intuitionistic fuzzy aggregation operators. *Journal of systems engineering and electronics*, *23*(4), 574–580. DOI:10.1109/JSEE.2012.00071

[31] Sudharsan, S., & .Ezhilmaran, D. (2014). Two new operator defined over interval valued intuitionistic fuzzy sets. *International journal of fuzzy logic systems*, *4*(4), 1–13. DOI:10.5121/ijfls.2014.4401

[32] Sudharsan, S., & Ezhilmaran, D. (2015). Some Operators Defined Over Intuitionistic Fuzzy Sets And Interval Valued Intuitionistic Fuzzy Sets. *International journal of mathematical archive eissn 2229-5046*, *6*(10), 30.

[33] Jamkhaneh, E. B. (2016). New operations over generalized interval valued intuitionistic fuzzy sets. *Gazi university journal of science*, *29*(3), 667–674.

[34] Mahmood, T., Abdullah, S., & Bilal, M. (2017). Multicriteria decision making based on cubic set. *Journal of new theory*, (16), 1-9. https://dergipark.org.tr/en/pub/jnt/issue/34515/381378

[35] Tarsuslu, S., Citil, M., Demirbas, E., & Aydın, M. (2017). Some modal operators with intuitionistic fuzzy sets. *Notes on intuitionistic fuzzy sets*, *23*(4), 20-28.

[36] Atanassov, K. T. (2004). On the Modal Operators defined over the intuitionistic Fuzzy Sets. *Notes on Intuitionistic fuzzy sets, 10*(1), 7-12. https://ifigenia.org/images/1/18/NIFS-10-1-07-12.pdf

[37] Kaur, G., & Garg, H. (2018). Cubic intuitionistic fuzzy aggregation operators. *International journal for uncertainty quantification*, *8*(5), 405–427. DOI:10.1615/Int.J.UncertaintyQuantification.2018020471

[38] Fitting, M. (2018). Necessity and possibility. In *Introduction to formal philosophy* (pp.323-331). DOI:10.1007/978-3-319-77434-3_15

[39] Chaira, T. (2019). *Operations on fuzzy/intuitionistic fuzzy sets and application in decision making*. Fuzzy set and its extension, 133–170. DOI:10.1002/9781119544203.ch5

[40] Rashid, S., Gulistan, M., Jun, Y. B., Khan, S., & Kadry, S. (2019). N-Cubic sets and aggregation operators. *Journal of intelligent and fuzzy systems*, *37*(4), 5009–5023. DOI:10.3233/JIFS-182595

[41] Wang, W., & Mendel, J. M. (2019). Interval-valued intuitionistic fuzzy aggregation methodology for decision making with a prioritization of criteria. *Iranian journal of fuzzy systems*, *16*(4), 115–127. (**In Persian**). https://ijfs.usb.ac.ir/article_4786_fb3e9dcbe788d8e40a8f7cf5ef0812f3.pdf

[42] Muneeza, & Abdullah, S. (2020). Multicriteria group decision-making for supplier selection based on intuitionistic cubic fuzzy aggregation operators. *International journal of fuzzy systems*, *22*(3), 810–823. DOI:10.1007/s40815-019-00768-x

[43] Atanassova, L., & Dworniczak, P. (2021). On the operation Δ over intuitionistic fuzzy sets. *Mathematics, 9*(13), 1518. DOI:10.3390/MATH9131518

[44] Fan, C., Fu, Q., Song, Y., Lu, Y., Li, W., & Zhu, X. (2022). A new model of interval-valued intuitionistic fuzzy weighted operators and their application in dynamic fusion target threat assessment. *Entropy*, *24*(12), 1825. DOI:10.3390/e24121825

[45] Abbasi Shureshjani, R., & Shakouri, B. (2021). A comment on "a novel parametric ranking method for intuitionistic fuzzy numbers". *Big data and computing visions*, *1*(3), 156–160. https://www.bidacv.com/article_142090.html

[46] Akram, M., Bashir, A., & Edalatpanah, S. A. (2021). A hybrid decision-making analysis under complex q-rung picture fuzzy Einstein averaging operators. *Computational and applied mathematics*, *40*(8), 1–35. DOI:10.1007/s40314-021-01651-y

[47] Dehghani Filabadi, A., & Hesamian, G. (2021). Development of a multi-period multi-attribute group decision-making method using type-2 fuzzy set of linguistic variables. *International journal of research in industrial engineering*, *10*(2), 138-154. DOI:10.22105/RIEJ.2021.272821.1187

[48] Akram, M., Shah, S. M. U., Al-Shamiri, M. M. A., & Edalatpanah, S. A. (2022). Fractional transportation problem under interval-valued Fermatean fuzzy sets. *AIMS math*, *7*(9), 17327-17348. DOI:10.3934/math.2022954

[49] Bhattacharya, J. (2022). Several significant equalities on intuitionistic fuzzy operators. *Notes on intuitionistic fuzzy sets*, *28*(2), 132–148. DOI:10.7546/nifs.2022.28.2.132-148

[50] Bhattacharya, J. (2023). Intuitionistic fuzzy operations and operators: Some observations. *Notes on intuitionistic fuzzy sets*, *29*(4), 371–382. DOI:10.7546/nifs.2023.29.4.371-382

[51] Bhattacharya, J. (2023). Some new results on intuitionistic fuzzy operators. *Notes on intuitionistic fuzzy sets*, *29*(3), 247–260. DOI:10.7546/nifs.2023.29.3.247-260

[52] Liu, Y., Yang, Z., He, J., Yu, L., & Zhong, Y. (2023). Some intuitionistic cubic fuzzy muirhead mean operators with their application to multicriteria decision making. *International journal of intelligent systems, 2023(1), 9891355*. DOI:10.1155/2023/9891355

[53] Nagarajan, D., Kanchana, A., Jacob, K., Kausar, N., Edalatpanah, S. A., & Shah, M. A. (2023). A novel approach based on neutrosophic Bonferroni mean operator of trapezoidal and triangular neutrosophic interval environments in multi-attribute group decision making. *Scientific reports,13(1), 10455*. DOI:10.1038/s41598-023-37497-z

[54] Adak, A. K., Kumar, D., & Edalatpanah, S. A. (2024). Some new operations on Pythagorean fuzzy sets. *Uncertainty discourse and applications*, *1*(1), 11-19.

[55] Ali, W., Shaheen, T., Haq, I. U., Smarandache, F., Toor, H. G., & Asif, F. (2024). An Innovative Approach on Yao's Three-Way Decision Model Using Intuitionistic Fuzzy Sets for Medical Diagnosis. *Neutrosophic systems with applications*, *18*, 1–13. DOI:10.61356/j.nswa.2024.18262

[56] Bakbak, D., Ulucay, V., & Edalatpanah, S. A. (2024). Trapezoidal fuzzy multi-number preference relations based on architecture multi-criteria decision-making application. *Iranian journal of fuzzy systems*, *21*(2), 51–65.

[57] Jameel, M., & Tanwar, S. (2024). A deeper monitoring and evaluation of the nature of barriers to climate change adaptation planning under fuzzy multi-criteria decision making methodology. *Multicriteria algorithms with applications*, *2*, 43-52. https://sciencesforce.com/index.php/mawa/article/view/71

[58] Mostafa, N. N., Kumar, A. K., & Ali, Y. (2024). A comparative study on x-ray image enhancement based on neutrosophic set. *Sustainable Machine Intelligence Journal*, *7*, 1–10. DOI:10.61356/smij.2024.77102

[59] Rasoulzadeh, M., Edalatpanah, S. A., Fallah, M., & Najafi, S. E. (2024). A hybrid model for choosing the optimal stock portfolio under intuitionistic fuzzy sets. *Iranian journal of fuzzy systems*, *21*(2), 161–179. DOI:10.22111/ijfs.2024.45118.7968

[60] Ghanizadeh, S. M. (2024). Human resource evaluation using a GRA-based decision model in an interval-valued fuzzy environment. *Journal of decisions and operations research*, *9*(1), 134-148.

[61] Akram, M., Ullah, I., Allahviranloo, T., & Edalatpanah, S. A. (2021). LR-type fully Pythagorean fuzzy linear programming problems with equality constraints. *Journal of intelligent and fuzzy systems*, *41*(1), 1975–1992. DOI:10.3233/JIFS-210655

[62] Alijanzadeh, M. R., Shayannia, S. A., & Movahedi, M. M. (2024). Optimization of maintenance in supply chain process and risk-based critical failure situations (case study: Iranian oil pipeline and telecommunication company, north district). *Journal of applied research on industrial engineering*, *11*(1), 125–142. DOI:10.22105/jarie.2022.322072.1419

[63] Uluçay, V., & Şahin, M. (2024). Intuitionistic fuzzy soft expert graphs with application. *Uncertainty discourse and applications, 1*(1), 1-10.

[64] Ur Rahman, A. (2023). A theoretical context for (θ,β)-convexity and (θ,β)-concavity with hypersoft settings. *Big data and computing visions*, *3*(4), 196–208. DOI:10.22105/bdcv.2023.192676

[65] Zare Ahmadabadi, H., Saffari Darberazi, A., Zamzam, F., Babakhanifard, M. S., Kiani, M., & Mofatehzadeh, E. (2024). A Model of the Factors Affecting Supply Chain Resilience: An Integrative Approach Incorporating Hesitant Fuzzy TOPSIS and Meta-Synthesis. *Journal of applied research on industrial engineering*, *11*(2), 195–211. DOI:10.22105/jarie.2024.433396.1592

[66] El-Morsy, S. (2023). Stock Portfolio Optimization Using Pythagorean Fuzzy Numbers. *Journal of operational and strategic analytics*, *1*(1), 8–13. DOI:10.56578/josa010205josa010205

[67] Faizi, S., Svitenko, H., Rashid, T., Zafar, S., & Sałabun, W. (2023). Some operations and properties of the cubic intuitionistic set with application in multi-criteria decision-making. *Mathematics*, *11*(5), 1190. DOI:10.3390/math11051190

[68] Gu, Y., Xu, K., Hu, X., & Chen, M. (2024). Research on improving the financial capacity of farmers based on fuzzy analytic hierarchy process. *Management analytics and social insights*, *1*(1), 88-102.

[69] Faghihmaleki, H., & Ghadiri, M. (2024). Operational strategies to the mechanical performance assessment of concrete with the neural network and artificial intelligence. *Innovation management & operational strategies*, *5*(1), 79-92. DOI: 10.22105/imos.2024.453210.1346

[70] Iqbal, W., Yang, T., & Ashraf, S. (2023). Optimizing Earthquake Response with Fermatean Probabilistic Hesitant Fuzzy Sets: A Decision Support Framework. *Journal of operational and strategic analytics*, *1*(4), 190–197. DOI:10.56578/josa010404

[71] Kiptum, C. K., Bouraima, M. B., Badi, I., Zonon, B. I. P., Ndiema, K. M., & Qiu, Y. (2023). Assessment of the challenges to urban sustainable development using an interval-valued fermatean fuzzy approach. *Systemic analytics*, *1*(1), 11–26. DOI:10.31181/sa1120233

[72] Mahalakshmi, P., Vimala, J., Jeevitha, K., & Nithya Sri, S. (2024). Advancing cybersecurity strategies for multinational corporations: novel distance measures in q-rung orthopair multi-fuzzy systems. *Journal of operational and strategic analytics*, *2*(1), 49–55. DOI:10.56578/josa020105

[73] Manoharan, P., Duraisamy, J., & Manoharan, S. (2024). Cardinality and relative cardinality on cubic intuitionistic fuzzy sets. *International Journal of Information Technology*, *16*, 4059–4068. DOI:10.1007/s41870-024-01876-0

[74] Alburaikan, A., Edalatpanah, S. A., Alharbi, R., & El-Wahed Khalifa, H. A. (2024). Towards neutrosophic Circumstances goal programming approach for solving multi-objective linear fractional programming problems. *International journal of neutrosophic science*, *23*(1), 350–365. DOI:10.54216/IJNS.230130

[75] Bodur, S., Topal, S., Gürkan, H., & Edalatpanah, S. A. (2024). A novel neutrosophic likert scale analysis of perceptions of organizational distributive justice via a score function: a complete statistical study and symmetry evidence using real-life survey data. *Symmetry*, *16*(5), 598. DOI:10.3390/sym16050598

[76] Christianto, V., & Smarandache, F. (2024). The enduring power of giving and caring from cavemen to kind hearts : a neutrosophic multi-criteria decision making approach. *Systems assessment and engineering management*, *1*, 26-33. https://doi.org/10.61356/j.saem.2024.1252

[77] Edalatpanah, S. A. (2020). Neutrosophic structured element. *Expert systems*, *37*(5), e12542. DOI:10.1111/exsy.12542

[78] Kumar, R., Edalatpanah, S. A., Jha, S., & Singh, R. (2019). A novel approach to solve gaussian valued neutrosophic shortest path problems. *International journal of engineering and advanced technology (IJEAT),* *8*(3), 347-353.

[79] Mohanta, K. K., & Sharanappa, D. S. (2023). Neutrosophic data envelopment analysis: a comprehensive review and current trends. *Optimality*, 1(1), 10-22.

[80] Garg, H., & Kaur, G. (2019). Cubic intuitionistic fuzzy sets and its fundamental properties. *Journal of multiple-valued logic and soft computing*, *33*(6), 507–537.